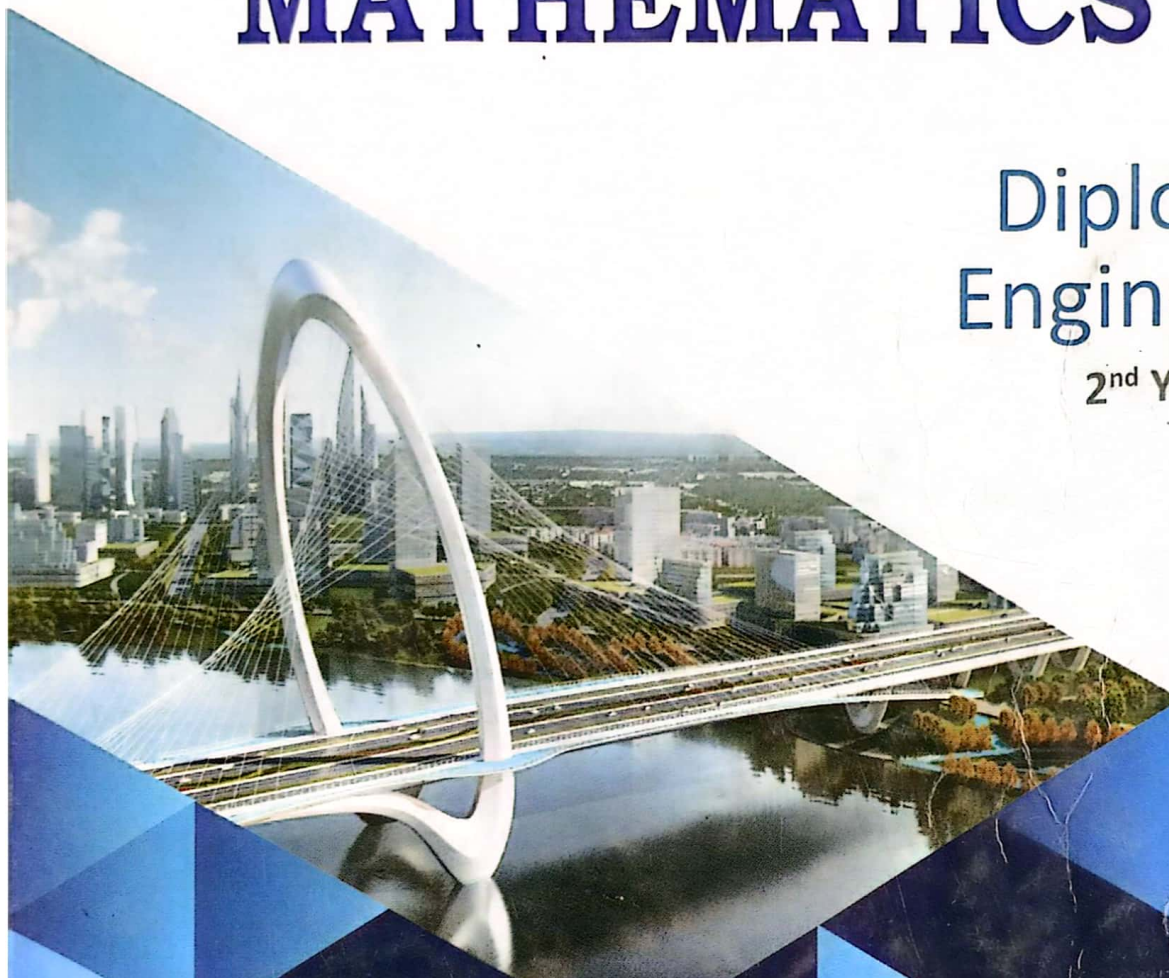


According to New Syllabus of
CTEVT 2021

ENGINEERING MATHEMATICS - III

for
Diploma in
Engineering
2nd Year (1st Part)



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PREFACE TO THE REVISED EDITION

We are pleasure to announce that the textbook entitled "**Engineering Mathematics- III for Diploma in Engineering**", has been possible to present as the second revised edition according to the new syllabus of CTEVT (2021). It would not be possible without the kind support, the valuable advice and suggestions from the readers, our colleague teachers and the tutors. We understand that the book has been widely accepted. Thus, at first, we express our sincere thanks to all the readers and the teachers those who preferred this book.

In this edition, the book is designed more effectively to meet the level and to fulfill the requirement of the syllabus prescribed by the Council for Technical Education and Vocational Training (CTEVT). We have tried our best to present the book in a systematic, easy and lucid way with appropriate examples. This book is organized in five units according to the new curriculum of CTEVT. The first unit is Application of Derivatives the second, third, fourth and fifth units are respectively Partial Derivatives, Application of Anti-derivatives, Differential Equations (ODES and PDES of first order) and Fourier series. All the topics, included according to the syllabus have been fully explained with suitable examples to illustrate them. Most of the examples are corrected and some of them have been added to make the book more effective and meaningful. The problems are divided into two parts (Difficulty level A and difficulty levels B) according to the new question pattern of this revised syllabus.

Again, we express our heartfelt thanks to Mr. Ananda Krishna Shrestha of Sukunda Pustak Bhawan, Bhotahity, Kathmandu for the active co-operation in republishing this book. We equally express our appreciation for the technical support of Zeeta Computer Service Pvt. Ltd.

We hope that the book will also be widely accepted and is found useful for the readers, teachers as well as to the tutors as it was before. We have tried our best to minimize any type of printing errors. However, we again heartily accept any suggestion and comments from the readers which will assist us to modify and improve for the upcoming publication.

Authors

PREFACE

This book **Engineering Mathematics for Diploma (Volume III)** is designed in order to cover the curriculum of the Diploma of Engineering (Second Year First Part) of CTEVT, government of Nepal. Moreover, it is helpful as a reference book to the students who are preparing for different competitive examinations for Bachelor of Engineering and equivalent courses. Furthermore, this book will be a good companion to the teachers who are teaching mathematics at diploma of engineering level. It is our best effort to make the prescribed course conceptually clear and comprehensive by systematically arranged subject matter. This textbook comprises theories, definitions and illustrative examples with a good collection of problems in the exercises, which instill the self-confidence on this students.

This book is organized in seven chapters as the curriculum prescribed for the different programs of diploma in engineering by the CTEVT. The chapters of this book include partial Derivatives, Differential Equations, Infinite Series, Fourier Series, Fourier Integral, Elementary Group Theory, and Analytical Solid Geometry. The matters of the book are presented in a simple and lucid way. A large number of worked out examples and problems in exercise have been arranged according to difficulty level. We hope this book will be fruitful for both students and teachers.

We would like to give our sincere thanks to Mr. Ananda Krishna Shrestha, **Sukunda Pustak Bhawan**, Bhotahity, Kathmandu for his active co-operation in publishing this book. We equally express our appreciation to GBG Computing for Word - processing and layout of the present the book in this form.

In this moment, we would like to acknowledge our respected senior teachers and our colleagues who support and encourage us to write this book. We have tried our best to minimize all kinds of printing errors. However, we heartily accept any suggestion and comments from the readers of this book which will assist us to modify and improve the upcoming publication.

Authors

SYLLABUS

Engineering Mathematics III

EGM 2104 SH

(All Programs)

Year: II

Lecture : 3 hours/week

Semester: I

Tutorial: 1 hour/week

Course Description: This course consists of five units namely : Applications of derivatives, partial derivatives, application of Anti-derivatives, Differential equations and Fourier series, which are basically necessary to develop mathematical knowledge and helpful for understanding as well as practicing their skills in the related engineering fields,

Course Objectives : On completion of this course, students will be able to understand the concept of the following topics and apply them in the related fields of different engineering areas: Applications of derivatives and anti-derivatives, partial derivatives, differential equations and the Fourier series.

Unit 1: Applications of Derivatives

12 hrs

- 1.1 Derivatives of inverse circular functions and hyperbolic functions
- 1.2 Differentials, tangent and normal
- 1.3 Maxima and minima, concavity, increasing and decreasing functions
- 1.4 Rate measures
- 1.5 Indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$ and $\infty - \infty$, L, Hospital's Rule (without proof)

Unit 2: Partial Derivatives

6 hrs

- 2.1 Functions of more than two variables
- 2.2 Partial derivative From first principles
- 2.3 Partial derivatives of First and Higher orders
- 2.4 Euler's theorem for function of two variables
- 2.5 Partial derivatives of composite functions

Unit 3: Applications of Anti-derivatives

8 hrs

- 3.1 Standard Integrals, related numerical problems
- 3.2 **Basic idea of curve sketching :** odd and even functions, periodicity of a function, symmetry (about x-axis, y-axis and origin), monotonicity of a function, sketching graphs of polynomial, trigonometric, exponential, and logarithmic functions (simple cases only)
- 3.3 Area under a curve using limit of sum (without proof)
- 3.4 Area between two curves (without proof)
- 3.5 Area of closed curves (circle and ellipse only)

Unit 4: Differential Equation**14 hrs****4.1 Ordinary Differential Equations (ODEs)**

- Definitions ,order and degree of differential equation
- Differential equation of First order and First degree
- Variable separation and variable change methods
- Homogenous and liner differential equation of First Order
- Exact differential equation, condition of exactness.
- Simple applications of first order differential equations.

4.2 Partial Differential Equations (PDEs)

- Basic concepts, definition and formation
- General solution of linear PDEs of first order ($Pp + Qq = R$ from)

Unit5: Fourier series**5 hrs**

- 5,1 Periodic functions and fundamental period of periodic functions
- 5,2 Odd and even functions with their properties
- 5,3 Trigonometric series
- 5,4 Fourier series in an interval of period 2π (arbitrary range is not required)

Evaluation Scheme

- Internal Assessment :20
- Final Exam :80

Unit Wise Marks division for Final

S.N.	Units	Short questions (2 marks)	Long questions (4 marks)	Total Marks
1.	Application of Derivatives	$4 \times 2 = 8$	$3 \times 4 = 12$	20
2.	Partial Derivatives	$2 \times 2 = 4$	$2 \times 4 = 8$	12
3.	Applications of Anti derivatives	$3 \times 2 = 6$	$3 \times 4 = 12$	18
4.	Differential Equations	$4 \times 2 = 8$	$4 \times 4 = 16$	24
5.	Fourier Series	$1 \times 2 = 2$	$1 \times 4 = 4$	6
		$14 \times 2 = 28$	$13 \times 4 = 52$	80

Reference Books

1. Thapa et al ,Engineering Mathematics- III,Three Years Diploma), Sukunda pustak Bhawan ,Bhotahity ,Kathmandu, Nepal
2. Bajrachary et al, Basic Mathematics (Grade XI /XII), Sukunda Pustak Bhawan, Bhotahity kathmandu ,Nepal
3. Kryszig E, Advanced Engineering Mathematics, wile -Easter publication, New Delhi ,India
4. Nath et al, Engineering Mathematics- III, Vidhyarthi publisher & distributors ,kathmandu , Nepal
5. Other references selected by the related lecturer(s) from among the texts available in the market that meet the content of this subject.

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Application of Derivatives

1.1 Introduction

Previously, we had discussed that the derivative (dy/dx) gives the slope of a tangent at a desired point on a curve of a function $y = f(x)$ in an interval $I \in \mathbb{R}$. Moreover, it is used to identify the increasing or decreasing function (or to identify the positive or negative slope of a curve) in the given interval as well as the second order derivative helps to find the concavity and the point of inflection of a curve in different intervals. Again it can be used to find the equation of the tangent at a given point to the curve of a function $y = f(x)$. Not only that but it is used to measure the rate of change of a component with respect to the rate of change of the another component which affect in our practical life about which it will briefly be discussed with examples in the particular units.

We have the trigonometric functions are $y = \sin x$, $y = \cos x$, $y = \tan x$. The inverses of them are respectively $x = \sin^{-1}y$, $x = \cos^{-1}y$ and $x = \tan^{-1}y$ which are called the inverse circular functions in some restricted intervals $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $[0, \pi]$ and $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ respectively where the values are ± 1 for sine and cosine and $-\infty$ to ∞ for tangent functions. Here we find the derivative of inverse circular function and Hyperbolic functions also. The sine, cosine and tangent hyperbolic functions are the certain combinations of exponential functions e^x and e^{-x} which are used to solve certain engineering problems. The hyperbolic functions are related to a rectangular hyperbola. The six hyperbolic functions are defined as

$$\begin{aligned} \sin hx &= \frac{e^x - e^{-x}}{2}, & \cos hx &= \frac{e^x + e^{-x}}{2}, & \tan hx &= \frac{e^x - e^{-x}}{e^x + e^{-x}}, & \cot hx &= \frac{e^x + e^{-x}}{e^x - e^{-x}}, \\ \sec hx &= \frac{2}{e^x + e^{-x}}, & \operatorname{cosec} hx &= \frac{2}{e^x - e^{-x}}. \end{aligned}$$

We discuss maxima and minima and tangent and normal. Finally, in this chapter we also discuss the problems of in-determinant forms $\frac{0}{0}$, $\frac{\infty}{\infty}$ and $\infty - \infty$ using the L'Hospital's Rule which are widely discussed below with appropriate examples.

1.2 Derivatives of Inverse Circular Functions

1. Derivative of $\sin^{-1} x$:

Let $y = \sin^{-1} x$

$$\therefore x = \sin y$$

(1)

Differentiating both sides with respect to y , we get

$$\begin{aligned} \frac{dx}{dy} &= \frac{d}{dy}(\sin y) \\ &= \cos y \\ &= \sqrt{1 - \sin^2 y} \end{aligned}$$

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$$= \sqrt{1-x^2}$$

[Using (1)]

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

2. Derivative of $\cos^{-1} x$:

$$\text{Let } y = \cos^{-1} x$$

$$\therefore x = \cos y$$

Differentiating both sides with respect to y gives

$$\begin{aligned} \frac{dx}{dy} &= \frac{d}{dy}(\cos y) \\ &= -\sin y \\ &= -\sqrt{1-\cos^2 y} \\ &= -\sqrt{1-x^2} \end{aligned}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

3. Derivative of $\tan^{-1} x$:

$$\text{Let } y = \tan^{-1} x$$

$$\therefore x = \tan y$$

Differentiating both sides with respect to y gives

$$\begin{aligned} \frac{dx}{dy} &= \frac{d}{dy}(\tan y) \\ &= \sec^2 y \\ &= 1 + \tan^2 y \end{aligned}$$

$$\text{or, } \frac{dx}{dy} = 1 + x^2$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

4. Derivative of $\operatorname{cosec}^{-1} x$:

$$\text{Let } y = \operatorname{cosec}^{-1} x.$$

$$\therefore x = \operatorname{cosec} y$$

Differentiating both sides with respect to y gives

$$\begin{aligned} \frac{dx}{dy} &= \frac{d}{dy}(\operatorname{cosec} y) \\ &= -\operatorname{cosec} y \cot y \\ &= -\operatorname{cosec} y \sqrt{\operatorname{cosec}^2 y - 1} \\ &= -x \sqrt{x^2 - 1} \end{aligned}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{x\sqrt{x^2-1}}$$

Similarly as above it can be proved that

$$5. \quad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$6. \quad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

1.3 Derivatives of Hyperbolic Functions

Let x be any real number. Then we define hyperbolic functions as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

$$\operatorname{cosech} x = \frac{2}{e^x - e^{-x}}$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Notes: Some of the identities related to hyperbolic functions are :

$$\sinh(-x) = -\sinh x$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh(-x) = \cosh x$$

$$\cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1, \tanh^2 x + \operatorname{sech}^2 x = 1$$

$$\sinh(-x) = -\sinh x$$

$$\sinh^2 x + \cosh^2 x = \cosh 2x, \cot^2 x - \operatorname{cosec}^2 x = 1$$

$$\tanh(-x) = -\tanh x$$

$$2 \sinh x \cosh x = \sinh 2x$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y,$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y.$$

Proof of some of the identities are as below :

1. We have, $\sinh(-x) = -\sinh x$

$$\text{Proof: LHS} = \sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = -\sinh x$$

$$\therefore \sinh(-x) = -\sinh x.$$

2. $\sinh^2 x + \cosh^2 x = \cosh 2x$

$$\text{Proof: L.H.S.} = \sinh^2 x + \cosh^2 x$$

$$\begin{aligned} &= \left(\frac{e^x - e^{-x}}{2}\right)^2 + \left(\frac{e^x + e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} - 2 + e^{-2x} + e^{2x} + 2 + e^{-2x}}{4} \\ &= \frac{2(e^{2x} + e^{-2x})}{4} = \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x \end{aligned}$$

$$\therefore \sinh^2 x + \cosh^2 x = \cosh 2x.$$

3. Let $y = \sinh x$

$$= \frac{e^x - e^{-x}}{2}$$

Differentiating both sides with respect to x gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right) \\ &= \frac{1}{2} \left(\frac{de^x}{dx} - \frac{de^{-x}}{dx} \right) \end{aligned}$$

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$$= \frac{1}{2} [e^x - e^{-x} (-1)]$$

$$= \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\therefore \frac{d}{dx}(\sinh x) = \cosh x$$

2. $\frac{d}{dx}(\cosh x) = \sinh x$

3. Let $y = \tanh x = \frac{\sinh x}{\cosh x}$

Differentiating both sides with respect to x gives

$$\frac{dy}{dx} = \frac{\cosh x \frac{d}{dx}(\sinh x) - \sinh x \frac{d}{dx}(\cosh x)}{(\cosh x)^2}$$

$$= \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{\cosh^2 x}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$

$$= \frac{1}{\cosh^2 x} \quad [\because \cosh^2 x - \sinh^2 x = 1]$$

$$\therefore \frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

4. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

5. $\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$

6. $\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$

Example: 1

Find $\frac{dy}{dx}$ when $y = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$

Solution:

Here, $y = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$

Putting $x = \tan \theta$ gives

$$y = \tan^{-1} \left(\frac{2 \tan \theta}{1 - \tan^2 \theta} \right)$$

$$= \tan^{-1}(\tan 2\theta)$$

$$= 2\theta$$

or $y = 2 \tan^{-1} x \quad [\because \theta = \tan^{-1} x]$

Differentiating both sides with respect to x gives

$$\frac{dy}{dx} = 2 \frac{d}{dx}(\tan^{-1} x)$$

$$= 2 \cdot \frac{1}{1+x^2}$$

$$= \frac{2}{1+x^2}$$

Example: 2

If $y = \cos^{-1}(1 - 2x^2)$, find $\frac{dy}{dx}$.

Solution:

Here,

$$y = \cos^{-1}(1 - 2x^2)$$

Putting $x = \sin \theta$ gives

$$y = \cos^{-1}(1 - 2\sin^2 \theta)$$

$$= \cos^{-1} \cos 2\theta$$

$$= 2\theta$$

i.e., $y = 2\sin^{-1}x$

Differentiating both sides with respect to x gives

$$\frac{dy}{dx} = 2 \frac{d}{dx}(\sin^{-1}x)$$

$$= 2 \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{2}{\sqrt{1-x^2}}$$

Example: 3

Find the derivative of $\sec^{-1}(\tan x)$.

Solution:

Let $y = \sec^{-1}(\tan x)$

Differentiating both sides with respect to x gives

$$\frac{dy}{dx} = \frac{d}{dx}[\sec^{-1}(\tan x)]$$

$$= \frac{d \sec^{-1}(\tan x)}{d \tan x} \times \frac{d \tan x}{dx}$$

$$= \frac{1}{\tan x \sqrt{\tan^2 x - 1}} \cdot \sec^2 x \quad \left[\because \frac{d(\sec^{-1}x)}{dx} = \frac{1}{x \sqrt{x^2 - 1}} \right]$$

$$\therefore \frac{dy}{dx} = \frac{\sec^2 x}{\tan x \sqrt{\tan^2 x - 1}}$$

Example: 4

Find the derivative of $\tan^{-1}\left(\frac{\sin 2x}{1 + \cos 2x}\right)$.

Solution:

Let $y = \tan^{-1}\left(\frac{\sin 2x}{1 + \cos 2x}\right)$

$$\begin{aligned}
 &= \tan^{-1} \left(\frac{2 \sin x \cos x}{2 \cos^2 x} \right) \\
 &= \tan^{-1} \tan x \\
 &= x
 \end{aligned}$$

Differentiating both sides with respect to x gives

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} (x) \\
 &= 1
 \end{aligned}$$

Example: 5

Find the derivative of $\sinh^{-1}(\cosh x)$.

Solution:

Let $y = \sinh^{-1}(\cosh x)$.

Differentiating both sides with respect to x gives

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d \sinh^{-1}(\cosh x)}{dx} \\
 &= \frac{d \sinh^{-1}(\cosh x)}{d \cosh x} \cdot \frac{d \cosh x}{dx} \\
 \therefore \frac{dy}{dx} &= \cosh x (\cosh x) \cdot \sinh x
 \end{aligned}$$

Example: 6

Find $\frac{dy}{dx}$ when $y = e^{\cosh x}$

Solution:

Here, $y = e^{\cosh x}$

Differentiating both sides with respect to x gives

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d e^{\cosh x}}{dx} \\
 &= \frac{d e^{\cosh x}}{d(\cosh x)} \cdot \frac{d(\cosh x)}{dx} \\
 &= e^{\cosh x} \cdot \sinh x \\
 \therefore \frac{dy}{dx} &= e^{\cosh x} \cdot \sinh x
 \end{aligned}$$

Example: 7

Find the derivative of $\log \sinh \frac{x}{a}$.

Solution:

Let $y = \log \sinh \frac{x}{a}$.

Differentiating both sides with respect to x gives

$$\frac{dy}{dx} = \frac{d \left(\log \sinh \frac{x}{a} \right)}{dx}$$

$$\begin{aligned}
 &= \frac{d \log \sinh \frac{x}{a}}{d \sinh \frac{x}{a}} \cdot \frac{d \sinh \frac{x}{a}}{d \left(\frac{x}{a} \right)} \cdot \frac{d \left(\frac{x}{a} \right)}{dx} \\
 &= \frac{1}{\sinh \frac{x}{a}} \cdot \cosh \frac{x}{a} \cdot \frac{1}{a}
 \end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{1}{a} \coth \frac{x}{a}$$

Example: 8

Find the derivative of $x^{\cosh \frac{x}{a}}$.

Solution:

$$\text{Let } y = x^{\cosh \frac{x}{a}}.$$

Taking logarithms on both sides gives

$$\begin{aligned}
 \log y &= \log (x)^{\cosh \frac{x}{a}} \\
 &= \cosh \frac{x}{a} \cdot \log x
 \end{aligned}$$

Differentiating both sides with respect to x gives

$$\begin{aligned}
 \frac{d \log y}{dx} &= \frac{d}{dx} \left[\cosh \frac{x}{a} \cdot \log x \right] \\
 \text{or, } \frac{d \log y}{dy} \cdot \frac{dy}{dx} &= \cosh \frac{x}{a} \frac{d}{dx} (\log x) + \log x \frac{d}{dx} \left(\cosh \frac{x}{a} \right) \\
 \text{or, } \frac{1}{y} \frac{dy}{dx} &= \cosh \frac{x}{a} \cdot \frac{1}{x} + \log x \cdot \sinh \frac{x}{a} \cdot \frac{1}{a} \\
 \text{or, } \frac{dy}{dx} &= y \left(\frac{1}{x} \cosh \frac{x}{a} + \frac{1}{a} \log x \cdot \sinh \frac{x}{a} \right) \\
 \therefore \frac{dy}{dx} &= (x)^{\cosh \frac{x}{a}} \left(\frac{1}{x} \cosh \frac{x}{a} + \frac{1}{a} \log x \cdot \sinh \frac{x}{a} \right)
 \end{aligned}$$

EXERCISE 1(I)**Difficulty Level A**

Find the derivatives of:

1. $x^3 \cot^{-1} x$

2. $\cos^{-1} (4x^3 - 3x)$

3. $\cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$

4. $e^x \sin^{-1} x$

5. $\sec^{-1} \left(\frac{1}{\sqrt{1-x^2}} \right)$

6. $\log (\tanh x)$

7. $e^{\sinh x}$

Difficulty Level B

Find the derivatives of:

8. $\left(\cosh \frac{x}{a}\right) \log x$

9. $(\cosh x)^{\sin^{-1}x}$

10. $x^{\sinh \frac{x}{a}}$

Answers:

1. $3x^2 \cot^{-1}x - \frac{x^3}{1+x^2}$

2. $-\frac{3}{\sqrt{1-x^2}}$

3. $\frac{2}{1+x^2}$

4. $e^x \left(\frac{1}{\sqrt{1+x^2}} + \sin^{-1}x \right)$

5. $\frac{1}{\sqrt{1-x^2}}$

6. $2 \operatorname{cosech} 2x$

7. $e^{\sinh x} \cdot \cosh x$

8. $\left(\cosh \frac{x}{a}\right)^{\log x} \left[\frac{1}{a} \log x \tanh \frac{x}{a} + \frac{1}{x} \log \cosh x \right]$

9. $(\cosh x)^{\sin^{-1}x} \left[\sin^{-1}x \cdot \tanh x + \frac{\log \cosh x}{\sqrt{1-x^2}} \right]$

10. $x^{\sinh \frac{x}{a}} \left[\frac{1}{x} \sinh \frac{x}{a} + \frac{1}{a} \log x \cdot \cosh \frac{x}{a} \right]$

□□□

1.4 Differentials

Let us consider $y = f(x)$ be a function of x . Now the differential of independent variable x is an small (arbitrary) increment of x which is denoted by $dx = \Delta x$. Again, the differential of dependent variable y is given by $dy = f'(x)dx$ where $f'(x)$ is the derivative of $f(x)$. As x is independent variable, $dx = \Delta x$ but for the dependent variable y , in general, $dy \neq \Delta y$. Generally, $\Delta y = f(x + \Delta x) - f(x)$ and $dy = f'(x)dx$. In figure, let $A(x, y)$ and $B(x + \Delta x, y + \Delta y)$ be two points which are very near to each other on the curve given by $y = f(x)$.

$$\text{Then, } f'(x) = \frac{\tan \theta}{1} = \frac{DC}{AC} \Rightarrow f'(x) = \frac{DC}{AC}$$

or, $DC = AC f'(x) = dx f'(x)$, i.e., $dy = f'(x) dx$ [Since $DC = dy$ and $BC = \Delta y$]

The difference between Δy and dy can be made very small as taking Δx sufficiently small such that $\Delta y \rightarrow dy$. Here, dy is the tangential increment and that of Δy is the curvilinear increment (Curve increment) for Δx or dx in x . For example:

Let $y = 2x^2$. Then $\Delta y = 2(x + \Delta x)^2 - 2x^2 = 2x^2 + 4x \cdot \Delta x + 2(\Delta x)^2 - 2x^2 = 2\Delta x(2x + \Delta x)$

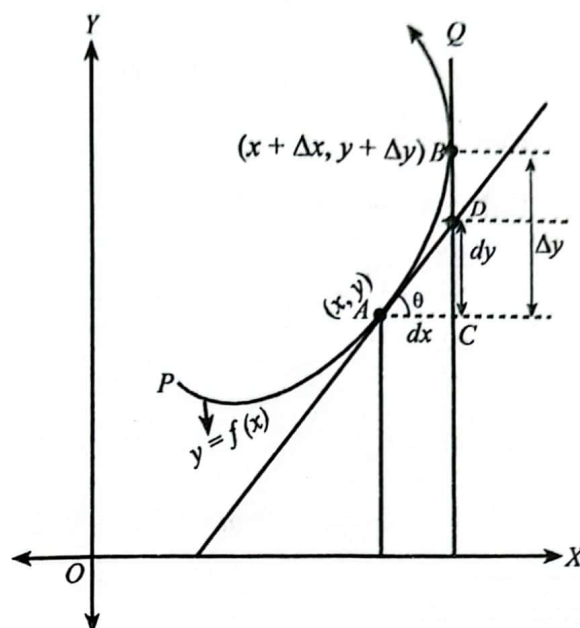
Again, $dy = f'(x) dx = 4x dx = 4x \Delta x$ ($\because dx = \Delta x$)

Now, if the side (x) of a square is changed from 3 to 3.02 cm then

$$\begin{aligned} \Delta x = dx &= 3.02 - 3 = 0.02, \Delta y = 2\Delta x(2x + \Delta x) \\ &= 2(0.02)(2 \times 3 + 0.02) \\ &= 0.04(6.02) = 0.2408 \text{ actual change in area} \end{aligned}$$

Again, $dy = 4(3)(0.02) = 12(0.02) = 0.24$, the approximate change in area

$$\begin{aligned} \text{Now, error} &= \Delta y - dy \\ &= 0.2408 - 0.24 \\ &= 0.0008 \end{aligned}$$



Note:

- (i) $dx = \Delta x$ but $dy \neq \Delta y$
- (ii) Approximate increase in y is $dy = f'(x) dx$
- (iii) Actual increase in y is $\Delta y = f(x + \Delta x) - f(x)$
- (iv) Error % = $\frac{\text{Error}}{\Delta y} \times 100\%$

Example: 1

Find an approximate change in $\frac{1}{x}$ as x changes from 1 to 0.98.

Solution:

Given, $x = 1$, $\Delta x = dx = 0.98 - 1 = -0.02$

Now, $y = \frac{1}{x}$

Differentiating with respect to 'x'

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{1}{x} \right)$$

$$\text{or, } \frac{dy}{dx} = -\frac{1}{x^2}$$

$$\therefore dy = -\frac{1}{x^2} dx$$

$$= -\frac{1}{(1)^2} \times (-0.02) = \frac{0.02}{1} = 0.02$$

Example: 2

The edge of a cube increase from 10 cm to 10.025cm. Find the approximate increments in the volume and the surface area of the cube. Also find the actual increments and the percentage error in the approximation.

Solution:

Here, Side of a cube = x where

$$x = 10 \text{ and } \Delta x = dx = 10.025 - 10 = 0.025$$

We have, Volume (V) = x^3

Approximate increase in volume,

$$\text{i.e. } \frac{dV}{dx} = \frac{d}{dx} (x^3) = 3x^2$$

$$\text{or, } \frac{dV}{dx} = 3x^2$$

$$\text{or, } dV = 3x^2 dx$$

$$\text{or, } dV = 3(10)^2 \times 0.025$$

$$\therefore dV = 3 \times 100 \times 0.025 = 7.5$$

Actual increase in volume

$$V = x^3$$

$$\text{or, } V + \Delta V = (x + \Delta x)^3$$

$$\text{or, } \Delta V = (x + \Delta x)^3 - V$$

$$\text{or, } \Delta V = (10 + 0.025)^3 - x^3$$

$$\text{or, } \Delta V = (10.025)^3 - 10^3$$

$$\therefore \Delta V = 1007.5187 - 1000 = 7.5187$$

We know that

$$\text{error \%} = \frac{\text{error}}{\Delta y} \times 100\%$$

$$\begin{aligned}\text{Since, error} &= \Delta V - dV \\ &= 7.5187 - 7.5 = 0.0187\end{aligned}$$

$$\text{So, error \%} = \frac{0.0187}{10^3} \times 100\% = \frac{0.0187}{1000} \times 100\% = 0.00187\%$$

Second part,

The surface area of the cube

$$S = 6x^2$$

Approximate increase in surface area

$$\frac{dS}{dx} = \frac{d}{dx}(6x^2)$$

$$\text{or, } \frac{dS}{dx} = 6(2x)$$

$$\therefore dS = 6 \times 2x dx = 6 \times 2 \times 10 \times 0.025$$

$$\therefore dS = 3$$

Actual increase in surface are

$$S = 6x^2$$

$$\text{or, } S + \Delta S = 6(x + \Delta x)^2$$

$$\text{or, } \Delta S = 6(x + \Delta x)^2 - S$$

$$\text{or, } \Delta S = 6(x + \Delta x)^2 - 6x^2$$

$$\text{or, } \Delta S = 6(10 + 0.025)^2 - 6 \times 10^2$$

$$\text{or, } \Delta S = 6 \times (10.025)^2 - 6 \times 100$$

$$\text{or, } \Delta S = 6 \times 100.50625 - 600$$

$$\text{or, } \Delta S = 603.00375 - 600$$

$$\text{or, } \Delta S = 3.00375$$

$$\text{error} = \Delta S - dS = 3.00375 - 3 = 0.00375$$

$$\text{and error \%} = \frac{0.00375}{6 \times 10^2} \times 100\% = \frac{0.00375}{600} \times 100\%$$

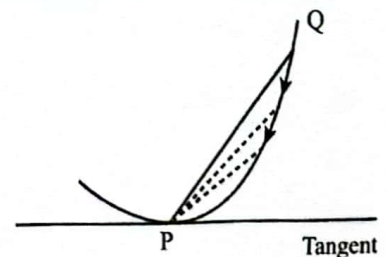
$$\text{error \%} = 0.000625\%$$

1.5 Tangent and Normal

The tangent at a point P to a given curve $y = f(x)$ is defined as the limiting position of the secant PQ as the point Q approaches P along the curve.

Thus, the tangent is a straight line which just touches the curve at a given point.

In mathematics, the word normal has a very specific meaning. It means **perpendicular** or at right angles. A straight line perpendicular to



the tangent and passing through the point of tangency is called the **normal** to the curve.

If α is the angle which is the positive direction of the tangent at P makes with the X-axis, then

$$\tan \alpha = m = \frac{dy}{dx}$$

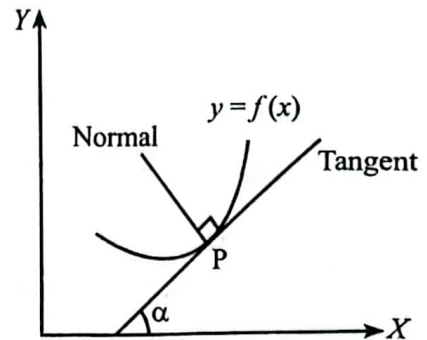
i.e. Slope of tangent line (m) = $\frac{dy}{dx}$.

Also, since normal is perpendicular to the tangent,

$$\text{Slope of tangent} \times \text{Slope of normal} = -1$$

$$\frac{dy}{dx} \times \text{Slope of normal} = -1$$

$$\therefore \text{Slope of normal} = -\frac{dx}{dy}$$



Note : $\tan \alpha = \frac{dy}{dx}$ is also known as the gradient of the curve at the point $P(x, y)$.

1.6 Equation of Tangent and Normal

Let $y = f(x)$ is a function which has finite derivative $f'(x)$ at $x = x_1$. The curve of a function $y = f(x)$ may have a tangent at $P(x_1, y_1)$ whose gradient (or slope) is given by $m = \tan \theta = \left(\frac{dy}{dx}\right)_{\text{at } x = x_1}$. Now the equation of the tangent at the point (x_1, y_1) is given by, $y - y_1 = m(x - x_1)$.

Notes:

- (i) If the slope (m) = 0, the equation of the tangent will be

$$y - y_1 = 0(x - x_1) \Rightarrow y - y_1 = 0$$

$$\Rightarrow y = y_1 \text{ i.e., a line parallel to } x\text{-axis.}$$

- (ii) If the slope (m) = $\tan \theta = \infty$ (i.e. $\theta = 90^\circ$ or 270°). Then the equation of the tangent will be

$$y - y_1 = \infty (x - x_1)$$

$$\Rightarrow x - x_1 = \frac{1}{\infty} (y - y_1) = 0$$

$$\Rightarrow x = x_1 \text{ i.e., a line parallel to the } y\text{-axis.}$$

- (iii) The tangent at $P(x, y)$ is parallel to the X-axis if $\alpha = 0$

$$\text{i.e., } \tan \alpha = 0 \Rightarrow \frac{dy}{dx} = 0$$

Such a tangent is also known as **Horizontal tangent**.

Thus, the vertical tangent at $P(x_1, y_1)$ is $y - y_1 = 0$.

- (iv) The tangent at $P(x, y)$ is perpendicular to the X-axis i.e., parallel to the Y-axis if $\alpha = \frac{\pi}{2}$

$$\text{i.e., } \tan \alpha = \tan \frac{\pi}{2} = \infty$$

$$\frac{dy}{dx} = \infty$$

$$\Rightarrow \frac{dx}{dy} = 0$$

Such a tangent is also known as **Vertical tangent**. Thus, the equation of vertical tangent at $P(x_1, y_1)$ is $x - x_1 = 0$.

Example: 3

Find the slope of the curve $y = 2x - x^2$ at (a) (0, 0) (b) (2, 1). Also find the point on the curve where the tangent is parallel to the x-axis.

Solution:

Here, the given curve is

$$y = 2x - x^2 \quad (1)$$

Differentiating with respect to x on both sides we have,

$$\frac{dy}{dx} = 2 - 2x.$$

Now,

$$(a) \text{ Slope at } (0, 0) \text{ is } \frac{dy}{dx} = 2 - 2 \times 0 = 2 \quad \therefore \frac{dy}{dx} = 2$$

$$(b) \text{ Slope at } (2, 1) \text{ is } \frac{dy}{dx} = 2 - 2 \times 2 = 2 - 4 = -2 \quad \therefore \frac{dy}{dx} = -2$$

For the tangent parallel to x-axis,

$$\frac{dy}{dx} = 0$$

$$\text{or, } 2 - 2x = 0 \Rightarrow 2x = 2$$

$$\therefore \text{ When } x = 1, y = 2 \times 1 - (1)^2 = 2 - 1 = 1 \quad [\text{From (1)}]$$

$\therefore$ The point at where the tangent is parallel to the x-axis is (1, 1).

Example: 4

Find the equation of tangent on the curve $f(x) = x^3 - 3x^2 + x - 1$ at point (3, 2).

Solution:

We have the curve

$$y = f(x) = x^3 - 3x^2 + x - 1$$

Differentiating y with respect to x ,

$$\frac{dy}{dx} = \frac{d}{dx} (x^3 - 3x^2 + x - 1) = 3x^2 - 6x + 1$$

$$\frac{dy}{dx} \text{ at } (3, 2)$$

$$\frac{dy}{dx} = 3(3)^2 - 6(3) + 1$$

$$= 27 - 18 + 1 = 10$$

Therefore, the equation of tangent at point (3, 2) is given by

$$y - y_1 = \frac{dy}{dx} (x - x_1)$$

$$\text{i.e., } y - 2 = 10(x - 3)$$

$$\text{i.e., } y - 2 = 10x - 30$$

$$\therefore y = 10x - 28$$

Which is the required equation of tangent.

Example: 5

Find the equation of normal to the curve $f(x) = x^2 + 3x + 1$ at $(0, 1)$.

Solution:

We have the curve

$$y = f(x) = x^2 + 3x + 1$$

Differentiating both sides with respect to x ,

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 3x + 1) = 2x + 3$$

At $(0, 1)$.

$$\frac{dy}{dx} = 2 \cdot 0 + 3 = 3$$

Now, the equation of normal at point $(0, 1)$ is given by

$$\frac{dy}{dx}(y - y_1) + (x - x_1) = 0$$

$$\text{i.e., } 3(y - 1) + (x - 0) = 0$$

$$\text{i.e., } 3y - 3 + x = 0$$

$$\therefore x + 3y = 3$$

Which is the required equation of normal.

Example: 6

Find the points on the curve $y = x^3 - 6x^2 + x + 3$ where the tangents are parallel to the line $y = x + 5$

Solution:

We have the given curve

$$y = x^3 - 6x^2 + x + 3 \quad (1)$$

Differentiating both sides with respect to x ,

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 6x^2 + x + 3) = 3x^2 - 12x + 1$$

Now, we have the line

$$y = x + 5$$

$$\therefore \text{Slope of line} = 1$$

Since the tangents are parallel to the line, then they must have same gradient.

$$\therefore \frac{dy}{dx} = 1$$

$$\text{i.e., } 1 = 3x^2 - 12x + 1$$

$$\text{i.e., } 3x^2 - 12x = 0$$

$$\text{i.e., } 3(x^2 - 4x) = 0$$

$$\text{i.e., } x^2 - 4x = 0$$

$$\text{i.e., } x(x - 4) = 0$$

$$\text{Either, } x = 0 \text{ or, } x = 4$$

Substituting the values of x in (1) to calculate the corresponding y coordinates.

When, $x = 0$,

$$\begin{aligned} y &= 0^3 - 6 \cdot 0^2 + 0 + 3 \\ &= 3 \end{aligned}$$

When, $x = 4$,

$$\begin{aligned} y &= 4^3 - 6 \cdot 4^2 + 4 + 3 \\ &= 64 - 96 + 4 + 3 \\ &= -25 \end{aligned}$$

So the required points are $(0, 3)$ and $(4, -25)$

Example: 7

Find the equation of tangents on the curve $f(x) = x^3 + x^2 + x + 1$ which is perpendicular to the line $2y + x + 5 = 0$

Solution:

We have the given curve

$$y = f(x) = x^3 + x^2 + x + 1 \quad (1)$$

Differentiating both sides with respect to x ,

$$\frac{dy}{dx} = \frac{d}{dx} f(x) = \frac{d}{dx} (x^3 + x^2 + x + 1) = 3x^2 + 2x + 1$$

Now, we have the line

$$2y + x + 5 = 0$$

$$\Rightarrow 2y = -x - 5$$

$$\Rightarrow y = -\frac{1}{2}x - \frac{5}{2}$$

$$\therefore \text{Slope of line} = -\frac{1}{2}$$

Since the tangents are perpendicular to the line, then the product of their slopes are -1 .

$$\text{i.e., } \frac{dy}{dx} \times \left(-\frac{1}{2}\right) = -1$$

$$\therefore \frac{dy}{dx} = 2$$

$$\text{So, } 2 = 3x^2 + 2x + 1$$

$$\text{i.e., } 3x^2 + 2x - 1 = 0$$

$$\text{i.e., } 3x^2 + 3x - x - 1 = 0$$

$$\text{i.e., } 3x(x+1) - 1(x+1) = 0$$

$$\text{i.e., } (x+1)(3x-1) = 0$$

$$\text{Either, } x+1 = 0 \Rightarrow x = -1$$

$$\text{or, } 3x-1 = 0 \Rightarrow x = \frac{1}{3}$$

Substituting the values of x in (1) to calculate the corresponding y coordinates.

$$\begin{aligned} \text{When, } x = -1, \quad y &= (-1)^3 + (-1)^2 + (-1) + 1 \\ &= -1 + 1 - 1 + 1 = 0 \end{aligned}$$

$$\begin{aligned}\text{When, } x = \frac{1}{3}, \quad y &= \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^2 + \frac{1}{3} + 1 \\ &= \frac{1}{27} + \frac{1}{9} + \frac{1}{3} + 1 \\ &= \frac{1+3+9+27}{27} = \frac{40}{27}\end{aligned}$$

So the points are $(-1, 0)$ and $\left(\frac{1}{3}, \frac{40}{27}\right)$.

$$\begin{aligned}\text{At } (-1, 0), \quad \frac{dy}{dx} &= 3(-1)^2 + 2(-1) + 1 \\ &= 3 \cdot 1 - 2 + 1 = 2\end{aligned}$$

$$\begin{aligned}\text{At } \left(\frac{1}{3}, \frac{40}{27}\right), \quad \frac{dy}{dx} &= 3 \cdot \left(\frac{1}{3}\right)^2 + 2 \cdot \frac{1}{3} + 1 \\ &= 3 \cdot \frac{1}{9} + \frac{2}{3} + 1 = \frac{1+2+3}{3} = 2\end{aligned}$$

Now, the equation of tangent at $(-1, 0)$ is given by

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$\text{i.e., } y - 0 = 2(x + 1)$$

$$\text{i.e., } y = 2x + 2$$

$$\therefore 2x - y + 2 = 0$$

Also, the equation of tangent at $\left(\frac{1}{3}, \frac{40}{27}\right)$ is given by

$$y - 0 \cdot \frac{40}{27} = 2\left(x - \frac{1}{3}\right)$$

$$\text{i.e., } \frac{27y - 40}{27} = \frac{2}{3}(3x - 1)$$

$$\text{i.e., } 27y - 40 = 18(3x - 1)$$

$$\text{i.e., } 27y - 40 = 54x - 18$$

$$\text{i.e., } 54x - 27y + 22 = 0$$

Hence, $2x - y + 2 = 0$ and $54x - 27y + 22 = 0$ are required equations of tangents.

Example: 8

Show that the tangent to the curve $\left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 = 2$ at (a, b) is $\frac{x}{a} + \frac{y}{b} = 2$.

Solution:

We have the curve $\frac{x^3}{a^3} + \frac{y^3}{b^3} = 2$

Differentiating both sides with respect to x ,

$$\frac{d}{dx} \left(\frac{x^3}{a^3} + \frac{y^3}{b^3} \right) = \frac{d}{dx} (2)$$

$$\text{i.e., } \frac{d}{dx} \left(\frac{x^3}{a^3} \right) + \frac{dy^3}{dy} \cdot \frac{dy}{dx} \cdot \frac{1}{b^3} = 0$$

$$\text{i.e., } \frac{1}{a^3} \cdot 3x^2 + 3y^2 \cdot \frac{1}{b^3} \frac{dy}{dx} = 0$$

$$\text{i.e., } 3y^2 \cdot \frac{1}{b^3} \frac{dy}{dx} = -\frac{3x^2}{a^3}$$

$$\text{i.e., } \frac{dy}{dx} = -\frac{b^3}{a^3} \cdot \frac{x^2}{y^2}$$

$$\text{At } (x, y) = (a, b), \quad \frac{dy}{dx} = -\frac{b^3}{a^3} \cdot \frac{a^2}{b^2} = -\frac{b}{a}$$

Now, the equation of the tangent to the curve at (a, b) is given by

$$y - b = -\frac{b}{a}(x - a)$$

$$\text{i.e., } ay - ab = -bx + ab$$

$$\text{i.e., } bx + ay = 2ab$$

Dividing both sides by ab ,

$$\frac{x}{a} + \frac{y}{b} = 2$$

Hence, proved.

Example: 9

Find the equations of the horizontal and vertical tangents to the curve $x^2 - xy + y^2 = 27$.

Solution:

We have the given curve $x^2 - xy + y^2 = 27$

(1)

Differentiating both sides with respect to x ,

$$\frac{d}{dx} x^2 - \frac{d}{dx} (xy) + \frac{dy^2}{dy} \cdot \frac{dy}{dx} = \frac{d}{dx} (27)$$

$$\text{or, } 2x - \left(x \frac{dy}{dx} + y \right) + 2y \cdot \frac{dy}{dx} = 0$$

$$\text{or, } 2x - x \frac{dy}{dx} - y + 2y \frac{dy}{dx} = 0$$

$$\text{or, } (2y - x) \frac{dy}{dx} = y - 2x$$

$$\therefore \frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

For horizontal tangent, we have

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{y - 2x}{2y - x} = 0$$

$$\Rightarrow y - 2x = 0$$

(2)

Solving (1) and (2)

$$x^2 - x(2x) + (2x)^2 = 27$$

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$$\text{or, } 3x^2 = 27$$

$$\therefore x = \pm 3$$

From equation (2), we get $y = \pm 6$

$$y = \pm 6$$

Hence the horizontal tangents at $(3, 6)$ and $(-3, -6)$ are

$$y - 6 = 0(x - 3) \text{ and } y - (-6) = 0(x + 3)$$

$$\text{i.e., } y - 6 = 0 \quad \text{and } y + 6 = 0$$

are required equations of horizontal tangents.

Again, for vertical tangent, we have

$$\frac{1}{\frac{dy}{dx}} = 0 \Rightarrow \frac{dx}{dy} = 0$$

$$\text{So, } \frac{2y - x}{y - 2x} = 0 \Rightarrow 2y - x = 0$$

$$\therefore x = 2y$$

(3)

Solving (1) and (3), we get $x = \pm 6$

$$(2y)^2 - 2y \cdot y + y^2 = 27$$

$$\text{or, } 3y^2 = 27$$

$$\therefore y = \pm 3$$

From equation (3)

$$x = \pm 6$$

Hence the vertical tangents at $(6, 3)$ and $(-6, -3)$ are $x = -6$ and $x = 6$.

Example: 10

Find the equations of the tangent and normal to the curve given by $x^2 - y^2 = 7$ at $(1, 2)$.

Solution:

Here, the given curve is

$$x^2 - y^2 = 7$$

Differentiating with respect to x on both sides we have,

$$2x - 2y \frac{dy}{dx} = 0$$

$$\text{or, } -2y \frac{dy}{dx} = -2x$$

$$\text{or, } \frac{dy}{dx} = \frac{x}{y}$$

$\therefore$ Slope of the tangent at $(1, 2)$ is

$$\frac{dy}{dx} = m = \frac{1}{2}$$

$\therefore$ The equation of the tangent at $(1, 2)$ is given by

$$y - y_1 = m(x - x_1)$$

$$\text{or, } y - 2 = \frac{1}{2}(x - 1)$$

$$\text{or, } 2y - 4 = x - 1$$

$$\text{or, } x - 2y = -4 + 1 = -3$$

$$\text{or, } x - 2y + 3 = 0$$

Again, the equation of the normal at (1, 2) is the line through (1, 2) is given by

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$\text{or, } y - 2 = \frac{1}{-1/2}(x - 1)$$

$$\text{or, } y - 2 = -2(x - 1) = -2x + 2$$

$$\text{or, } 2x + y = 4$$

$\therefore$ The required equation of the tangent at (1, 2) is $x - 2y + 3 = 0$

and the equation of the normal at the point is $2x + y = 4$.

1.7 Angle between Two Intersecting Curves

If two curves intersect, the angle between them is defined by the angle between the tangents to the curves at the point of intersection.

The process of finding (or obtaining) the angle between two intersecting curves.

- At first, solve the given equation of curves to get the point of intersections.
- Secondly, find the slopes of the tangents say m_1 and m_2 at the point of intersections.

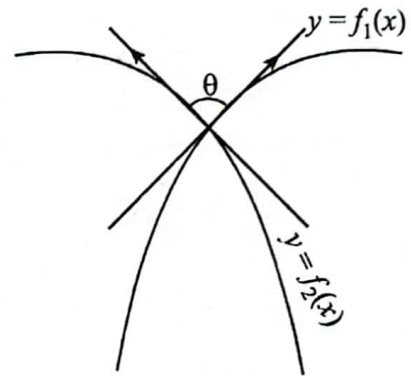
Then,

Case-I: If $m_1 = m_2$, the angle of intersection is $\phi = 0^\circ$.

Case - II: If $m_1 = -\frac{1}{m_2}$, the angle of intersection is $\phi = 90^\circ$ otherwise

$$\tan \phi = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\text{i.e., } \phi = \tan^{-1} \left(\frac{m_1 - m_2}{1 + m_1 m_2} \right)$$



Example: 11

Find the angle of intersection of the parabolas $y^2 = 4x$ and $x^2 = 4y$ at their point of intersection other than the origin.

Solution:

We have the given curve

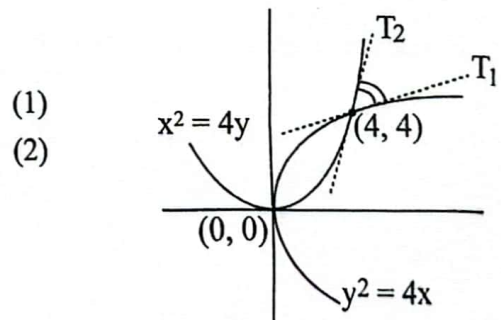
$$y^2 = 4x \quad (1)$$

$$\text{and } x^2 = 4y \quad (2)$$

Solving (1) and (2), we get

$$\left(\frac{x^2}{4} \right)^2 = 4x$$

$$\Rightarrow x^4 = 64x$$



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$$\Rightarrow x^4 - 64x = 0$$

$$\Rightarrow x(x^3 - 64) = 0$$

Either $x = 0$

$$\text{or, } x^3 - 64 = 0 \Rightarrow x^3 = 4^3$$

$$\Rightarrow x = 4$$

When $x = 0$, $y = 0$

When $x = 4$, $y = 4$

Here, (0, 0) and (4, 4) are the two points of intersection of the parabolas.

Now, Differentiating (1) with respect to x ,

$$\frac{dy^2}{dy} \cdot \frac{dy}{dx} = \frac{d}{dx}(4x)$$

$$\Rightarrow 2y \frac{dy}{dx} = 4$$

$$\frac{dy}{dx} = \frac{2}{y}$$

$$\text{At } (4, 4), \frac{dy}{dx} = \frac{2}{4} = \frac{1}{2} \text{ (} m_1 \text{ say)}$$

Again, Differentiating (2) with respect to x ,

$$4 \frac{dy}{dx} = \frac{d}{dx}(x^2) = 2x$$

$$\frac{dy}{dx} = \frac{x}{2}$$

$$\text{At } (4, 4), \frac{dy}{dx} = \frac{4}{2} = 2 \text{ (} m_2 \text{ say)}$$

Let θ be the angle between two parabolas, then

$$\begin{aligned} \theta &= \tan^{-1} \left(\pm \frac{m_1 - m_2}{1 + m_1 m_2} \right) \\ &= \tan^{-1} \left(\pm \frac{\frac{1}{2} - 2}{1 + \frac{1}{2} \cdot 2} \right) \\ &= \tan^{-1} \left(\pm \frac{\frac{3}{2}}{2} \right) = \tan^{-1} \left(\pm \frac{3}{4} \right) \end{aligned}$$

Hence the angle of intersection of two parabolas is $\theta = \tan^{-1} \pm \left(\frac{3}{4} \right)$.

EXERCISE - I (II)**Difficulty Level A**

- Find approximate change (dy) in x^3 as x change from 5 to 5.01.
- A circular copper plate is heated so that its radius increase from 5 cm to 5.06 cm. Find the approximate increase in area and also the actual increase in area.
- Find the approximate increase in the surface area of a cube if the edge increase from 10 to 10.01 cm. Also calculate the percentage error in the use of differential approximation.
- Find the approximate increase in the volume of a sphere when its radius increase from 2 to 2.1. Find also the actual increase and compare the two values.
- Compute Δy , dy and $\Delta y - dy$ when $y = \frac{x^2}{2} + 3x$ where, $x = 2$ and $dx = 0.5$ (or, x change from 2 to 2.5)
- Find the slope and the inclination with the x -axis of the tangent of the curves:
 - $2y = 2 - x^2$ at $x = 1$
 - $x^2 + y^2 = 25$ at $(-3, 4)$

Difficulty Level B

- Find the equation of tangent to the given curve at the points indicated
 - $f(x) = 3x^2 - 2x + 4$ at $(0, 4)$
 - $f(x) = 5x^3 + 12x^2 - 7x$ at $(-1, 14)$
- Find the equation of normal to the given curve at the points indicated.
 - $f(x) = x^2 + 3x + 1$ at $(4, 29)$
 - $f(x) = 2x^3 - 5x + 4$ at $(-1, 7)$
- Find the tangents and normal to the given curves at the given points:
 - $y^2 = 4ax$ at $(a, -2a)$
 - $x^2 + y^2 = 25$ at $(3, -4)$
 - $x^2 - y^2 = 7$ at $(4, -3)$
 - $y = x^3 - 2x^2 + 4$ at $(2, 4)$
- Find the points on the curve:
 - $y = x^3 - 3x^2 - 9x + 15$ where the tangent is parallel to the X -axis.
 - $y = 2x - x^2$ where the tangents are parallel to the x -axis
 - Find the point on the curve $y^2 = 4x + 1$ at which the tangent is perpendicular to the line $7x + 2y = 10$
- Find the equation of tangents to the curve $f(x) = x^3 - 5x^2 + 5x - 4$ which is parallel to the line $y = 2x + 1$.
- Find the equations of the horizontal and vertical tangents to the curve $y = (x - 2)(x - 3)^2$.
- Find the angles between the intersecting curves
 - $y = x^2$ and $x = y^2$.
 - $y^2 = x^3$ and $y = 2x$
- Find the angle of intersection of the circles $x^2 - 4x + y^2 = 0$ and $x^2 + y^2 = 8$.
- Find the points on the circle $x^2 + y^2 = 16$ at which the tangents are parallel to the
 - x -axis
 - y -axis.
- Show that the equation of the tangent with non-zero slope m to the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$

ANSWERS

1. 0.75
2. $0.6 \pi \text{ cm}^2$, $0.6036 \pi \text{ cm}^2$.
3. 1.2 cm^2 , 0.0001%
4. 1.6π , $\frac{5.044}{3} \pi$, 0.9516
5. 2.625, 2.5, 0.125
6. (a) $m = -1$, $\theta = 135^\circ$
- (b) $m = \frac{3}{4}$, $\theta = \tan^{-1} \left(\frac{3}{4} \right)$
7. (a) $y + 2x = 4$
- (b) $y + 16x + 2 = 0$
8. (a) $y - 16x + 23 = 0$
- (b) $y + 16x + 2 = 0$
9. (a) $x + y + a = 0$, $x - y = 3a$
- (b) $3x - 4y = 25$, $4x + 3y = 0$
- (c) $4x + 3y = 7$, $3x - 4y = 24$
- (d) $4x - y = 4$, $x + 4y = 18$
10. (a) $(-1, 20)$; $(3, -12)$
- (b) $(x, y) = (1, 1)$
- (c) $(x, y) = (12, 7)$
11. $27y - 54x + 95 = 0$, $y - 2x + 13 = 0$
12. $y = 0$, $27y - 4 = 0$
13. (a) $\theta = \tan^{-1} \left(\frac{2}{3} \right)$
- (b) $\theta = \tan^{-1} (2)$ at $(0, 0)$ and $\theta = \tan^{-1} \left(\frac{1}{7} \right)$ at $(4, 8)$
14. 45°
15. (a) $(0, \pm 4)$, (b) $(\pm 4, 0)$

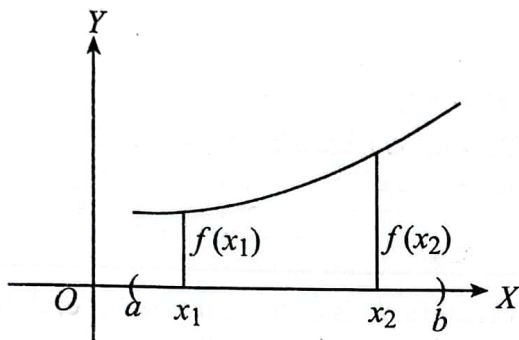
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1.8 Increasing and Decreasing Functions

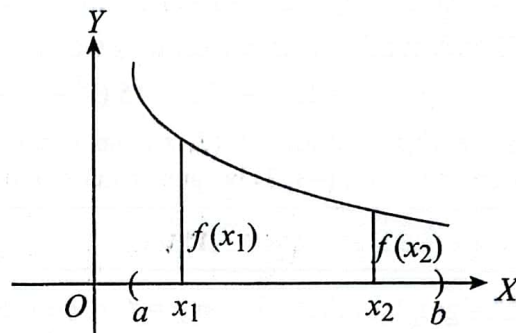
Derivatives has many applications. For example, rate measure, finding the nature of a functions and sketching their graph, find maximum and minimum value of a function, finding slope of tangent at a point etc. But here, we confine to find the maximum and minimum value of a function at a point. For this, we use derivative to find whether the function is increasing or decreasing in an interval and whether a function is concave upward or downward in an interval.

A function $f(x)$ is said to be increasing in an interval (a, b) if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$.

A function $f(x)$ is said to be decreasing in an interval (a, b) if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a, b)$.

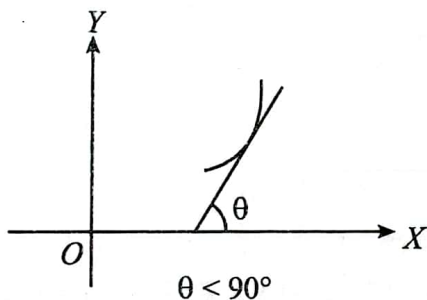


Increasing in (a, b)

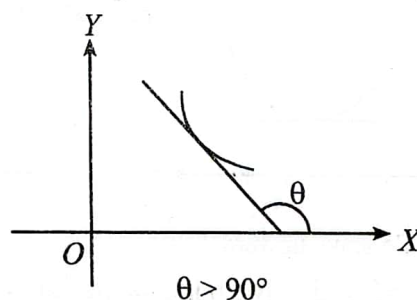


Decreasing in (a, b)

Here we use the derivative to test increasing and decreasing function



$\therefore \tan \theta$ is positive, i.e. $f'(x) > 0$

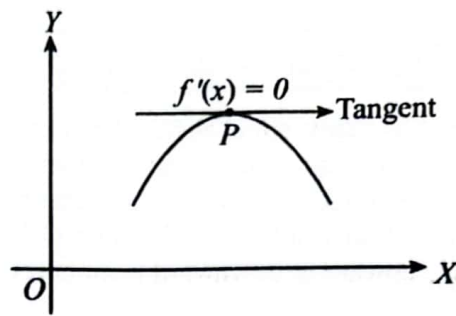
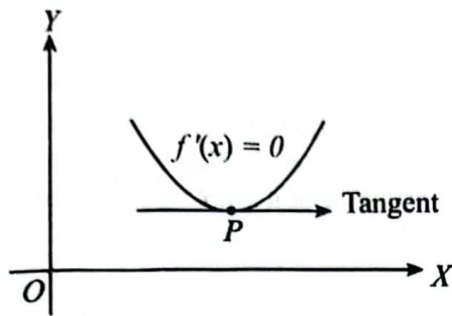


$\therefore \tan \theta$ is negative i.e. $f'(x) < 0$

- If $f'(x) > 0$ for all $x \in (a, b)$ then $f(x)$ is increasing on (a, b) , in this case slope of tangent in (a, b) is less than 90° and hence is positive.
- If $f'(x) < 0$ for all $x \in (a, b)$ then $f(x)$ is decreasing on (a, b) in this case the slope of the tangent in (a, b) is greater than 90° and hence is negative.

Critical Point (Stationary point)

If $f'(x) = 0$ at $x = a$ then the function is neither increasing nor decreasing. In this case the tangent of the curve at $x = a$ is parallel to x -axis. The point where $f'(x) = 0$ is called stationary point or critical point. In other words, a point which divides the increasing and decreasing of a function is called critical point. At this point tangent is parallel to the x - axis as shown in the figure.

**Example: 1**

Find the interval in which the function $f(x) = 5x^3 - 135x + 22$ is increasing or decreasing.

Solution:

We have, $f(x) = 5x^3 - 135x + 22$

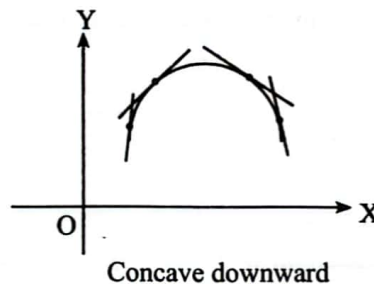
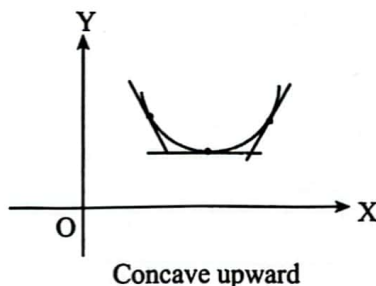
Differentiating with respect to x , we get

$$f'(x) = 15x^2 - 135 = 15(x^2 - 9) = 15(x - 3)(x + 3)$$

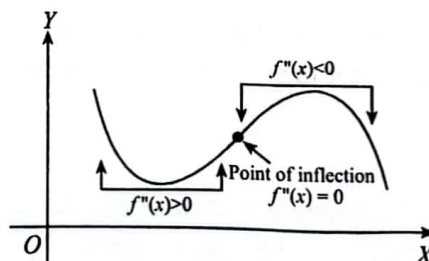
Here, $f'(x) > 0$ for $x \in (3, \infty)$ and $x \in (-\infty, -3)$. So, $f(x)$ is increasing on $(-\infty, -3) \cup (3, \infty)$ and decreasing on $(-3, 3)$ because $f'(x) < 0$ in the interval $(-3, 3)$.

1.9 Concavity of Curves

The graph of the function $y = f(x)$ defined in an interval (a, b) is concave upward if each point on the graph lies above any tangent to it in that interval. But if each point on the graph lies below any tangent to the curve in the interval, then the graph of the function is said to be concave downward.



The second derivative corresponds to the curvature or concavity of the graph. Here we use the second derivative to test concavity of curve. The sign of the second derivative gives us information about its concavity. If the second derivative of function $f(x)$ is defined on an interval (a, b) and $f''(x) > 0$ on this interval then the derivative of the derivative is positive. Thus the derivative is increasing and the graph is concave upward.



- If $f''(x) > 0$ in an interval (a, b) then $f(x)$ is concave upward in that interval.
- If $f''(x) < 0$ in an interval (a, b) then $f(x)$ is concave downward in that interval.

Point of Inflection

If $f''(x) = 0$ at point $x = a$ then this point separates the portion of the graph which is concave downwards from the portion which is concave upwards. Such a point is called point of inflection.

Example: 2

Determine where the graph of the function is concave upwards and where it is concave downward $f(x) = x^4 - 2x^3 + 5$. Also find the point of inflection.

Solution:

$$\text{Here, } f(x) = x^4 - 2x^3 + 5$$

$$f'(x) = 4x^3 - 6x^2$$

$$f''(x) = 12x^2 - 12x = 12x(x - 1)$$

Here, $f''(x) > 0$ in $(-\infty, 0)$ and $(1, \infty)$ so, $f(x)$ is concave upward in $(-\infty, 0) \cup (1, \infty)$. Also, $f(x)$ is concave downward in $(0, 1)$ because $f''(x) < 0$ in this interval. Also, for point of inflection,

$$f''(x) = 0 \quad \text{i.e., } 12x(x - 1) = 0$$

$$\text{Either } x = 0 \quad \text{or, } x - 1 = 0$$

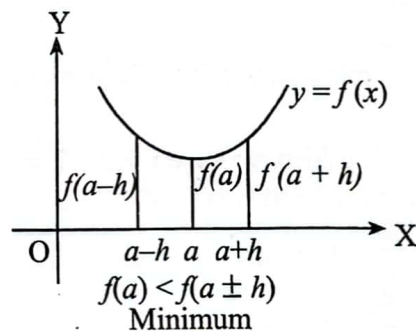
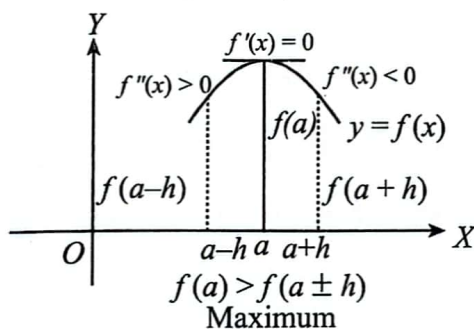
Hence, $x = 0$ and $x = 1$ are points of inflection.

1.10 Maxima and Minima

A function $y = f(x)$ is said to have maximum value or maxima at $x = a$ if $f(a) > f(a + h)$ and $f(a) > f(a - h)$ for small positive value of h .

A function $y = f(x)$ is said to have minimum value or minima at $x = a$ if $f(a) < f(a + h)$ and $f(a) < f(a - h)$ for small positive value of h .

The maximum and minimum values at some point in the curve $y = f(x)$ are called local maxima and local minima respectively.

**Use of Derivative for Maxima and Minima**

(a) If $f'(x) = 0$ and $f''(x) < 0$ at $x = a$ then the function $f(x)$ is maximum at $x = a$.

(b) If $f'(x) = 0$ and $f''(x) > 0$ at $x = a$ then the function $f(x)$ is minimum at $x = a$.

Steps to Find Maxima or Minima (Extreme Values) of the function $y = f(x)$

(1) Find $f'(x)$ and $f''(x)$.

(2) For maxima and minima, $f'(x) = 0$ and find the values of x .

(3) Put these values of x in $f''(x)$.

(4) (a) If $f''(x) < 0$ at $x = a$, then $f(x)$ is maximum at $x = a$ and the maximum value is $f(a)$.

(b) If $f''(x) > 0$ at $x = a$, then $f(x)$ is minimum at $x = a$ and the minimum value is $f(a)$.

(5) If $f'(x) \neq 0$ for any value of x , then there is no maxima or minima.

Example: 3

Find the maxima and minima and point of inflection of the function

$$f(x) = 2x^3 - 15x^2 + 36x + 5$$

[CTEVT 2063, 065 B, 066, 066B]

Solution:

$$\text{We have, } f(x) = 2x^3 - 15x^2 + 36x + 5$$

$$f'(x) = 6x^2 - 30x + 36$$

$$f''(x) = 12x - 30 = 6(2x - 5)$$

For maxima and minima

$$f'(x) = 0$$

$$\text{or, } 6x^2 - 30x + 36 = 0$$

$$\text{or, } x^2 - 5x + 6 = 0$$

$$\text{or, } x^2 - 3x - 2x + 6 = 0$$

$$(x - 3)(x - 2) = 0$$

Either $x = 3$ or $x = 2$ When $x = 2$, $f''(x) = 24 - 30 = -6 < 0$. So, $f(x)$ is maximum at $x = 2$ and the maximum value is

$$f(2) = 2 \times 2^3 - 15 \times 2^2 + 36 \times 2 + 5 = 16 - 60 + 72 + 5 = 33$$

When $x = 3$, $f''(x) = 36 - 30 = 6 > 0$. So, $f(x)$ is minimum at $x = 3$ and the minimum value is

$$f(3) = 2 \times 3^3 - 15 \times 3^2 + 36 \times 3 + 5 \\ = 54 - 135 + 108 + 5 = 32$$

For point of inflection, we have

$$f''(x) = 0 \text{ i.e., } 6(2x - 5) = 0$$

$$\therefore x = \frac{5}{2} \text{ is the point of inflection.}$$

Example: 4Find the minimum value of x^2y when $x + y = 7$.**Solution:**

$$\text{Let } u = x^2y \text{ Also we have } x + y = 7 \Rightarrow y = 7 - x$$

$$\text{Thus, } u = x^2(7 - x) = 7x^2 - x^3$$

$$\text{Now, } \frac{du}{dx} = 14x - 3x^2$$

$$\frac{d^2u}{dx^2} = 14 - 6x$$

For maxima and minima we must have, $\frac{du}{dx} = 0$.

$$\text{So, } 14x - 3x^2 = 0$$

$$\text{or, } x(14 - 3x) = 0$$

$$\text{Either } x = 0 \quad \text{or, } x = \frac{14}{3}$$

When $x = 0$, $\frac{d^2u}{dx^2} = 14 > 0$. So, u is minimum at $x = 0$ and the minimum value is

$$u(0) = 7 \cdot 0^2 - 0^3 = 0.$$

$$\text{When } x = \frac{14}{3}, \frac{d^2u}{dx^2} = 14 - 6 \times \frac{14}{3} = -14 < 0$$

So, u is maximum at $x = \frac{14}{3}$ and the maximum value is

$$u\left(\frac{14}{3}\right) = 7 \times \left(\frac{14}{3}\right)^2 - \left(\frac{14}{3}\right)^3 = \frac{1372}{27}$$

Example: 5

A gardener who has 144 ft of fencing wire wishes to enclose a rectangular garden. Find the maximum area he can enclose.

Solution:

Let x and y denotes the length and breadth of the rectangular garden.

Then, according to given condition.

$$2(x + y) = 144$$

$$x + y = 72$$

$$\therefore y = 72 - x$$

Let A be the area of the rectangular garden which is to be maximize.

$$\text{Then, } A = xy = x(72 - x) = 72x - x^2$$

Differentiating both sides with respect to x , we get

$$\frac{dA}{dx} = 72 - 2x$$

$$\text{Again, } \frac{d^2A}{dx^2} = -2 < 0$$

For maxima and minima

$$\frac{dA}{dx} = 0$$

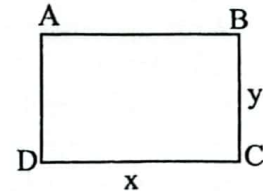
$$\therefore 72 - 2x = 0$$

$$\therefore x = 36$$

$$\text{Also, } y = 72 - x = 36$$

Again at $x = 36$, $\frac{d^2A}{dx^2} < 0$ so, A is maximum when $x = 36$.

The maximum area $= xy = 36 \times 36 = 1296$ sq.ft.



EXERCISE 1 (III)**Difficulty Level: A**

- Define increasing and decreasing functions.
- Define about point of stationery (i.e. critical points). In which condition does the point of inflections be obtained?
- Write the conditions to have a local (a) maxima (b) minima, of a function $f(x)$ at a certain point in some interval.
- Find the critical points of the functions below:**
 - $f(x) = x^3 - 3x^2 + 5$
 - $f(x) = 2x^3 - 15x^2 + 36x + 9$
 - $f(x) = x^4 - 4x^3 - 2x^2 + 12x + 7$
- Test whether the given functions below are increasing or decreasing in the given intervals.**
 - $2x^3 + 9x^2 + 12x + 20$ [(i) $x > -1$ (ii) $x < -2$ (iii) $-2 < x < -1$]
 - $2x^3 - 15x^2 + 36x + 5$ [(i) $x < 2$ (ii) $x > 3$ (iii) $2 < x < 3$]
- Find the point of inflection of the functions:**
 - $f(x) = x^3 - 12x + 8$
 - $f(x) = 2x^3 - 3x^2 + 36x + 1$
 - $f(x) = x + \frac{100}{x} - 1$

Difficulty Level: B

- Find the intervals in which the following functions are increasing or decreasing.**
 - $f(x) = 3x^2 - 6x + 5$
 - $f(x) = x^3 - 3x^2 + 1$
- Determine where the graph of following curves are concave upwards or concave downwards.**
 - $f(x) = x^3 - 3x^2 - 9x + 9$
 - $f(x) = x^4 - 2x^3 + 5$
 - $f(x) = x^4 - 6x^2 - 8x + 10$
- Find local maxima and minima of the following functions. Also find the point of inflection.**
 - $f(x) = x^3 - 3x^2 + 9$
 - $f(x) = 2x^3 - 15x^2 + 36x + 11$ [CTEVT 2073]
 - $f(x) = x^3 - 2x^2 + x + 6$
 - $f(x) = 4x^3 - 15x^2 + 12x + 7$
 - $f(x) = x^3 - 3x + 4$
 - $f(x) = x^4 - 14x^2 + 24x + 1$
- Find the maximum and minimum values of the function $f(x) = x^4 - 4x^3 - 2x^2 + 12x + 7$. Also find the point of inflection. [CTEVT 2068]
- Find the local maximum and minimum of the function $y = 4x^3 - 6x^2 - 9x + 1$. Also find the point of inflection. [CTEVT 2064]
- Find the minimum value of
 - $x^2 + y^2$ when $x + y = 6$
 - $x + y = a$
- Show that function $x^5 - 5x^4 + 5x^3 - 1$ is maximum when $x = 1$, minimum when $x = 3$ and neither when $x = 0$.
- Find the maximum area of rectangular plot of land which can be enclosed by a rope of length 60 m.

15. A wire of length 36 cm is to form a rectangle find the dimensions of rectangle so that it has the maximum area.
16. A gardener having 120 meters of fencing materials wishes to enclose a rectangular plot of land and also to erect of fence across the land parallel to two of its side. Find the maximum area he can enclose.

ANSWERS

- | | | |
|---|---|--|
| <p>4. (a) $x = 0, 2$</p> <p>5. (a) (i) Increasing
(b) (i) Increasing</p> <p>6. (a) $x = 0$</p> <p>7. (a) Increasing $(1, \infty)$, decreasing $(-\infty, 1)$
(b) Increasing $(-\infty, 0) \cup (2, \infty)$, decreasing $(0, 2)$</p> <p>8. (a) Concave upward for $x > 1$ and downward for $x < 1$.
(b) Concave upward for $x < 0$ and $x > 1$ and downward for $0 < x < 1$.
(c) Concave upward for $x > 1$ and $x < 1$ and downward for $-1 < x < 1$.</p> <p>9. (a) Maximum value is 9 at $x = 0$ minimum value is 5 at $x = 2$ point of inflection at $x = 1$
(b) Maximum value is 39 at $x = 2$, minimum value is 38 at $x = 3$, point of inflection at $x = \frac{5}{2}$
(c) Maximum value is $\frac{166}{27}$ at $x = \frac{1}{3}$, minimum value is 6 at $x = 1$, point of inflection at $x = \frac{2}{3}$
(d) Maximum value is $\frac{39}{4}$ at $x = \frac{1}{2}$, minimum value is 3 at $x = 2$, point of inflection at $x = \frac{5}{4}$
(e) Maximum value is 6 at $x = 1$, minimum value is 2 at $x = 1$, and point of inflection at $x = 0$
(f) Maximum value is 12 at $x = 1$, minimum value is 9 at $x = 2$, minimum value -116 at $x = -3$, point of inflection at $x = \pm\sqrt{7/3}$.</p> <p>10. Maximum value is 14 at $x = 1$, minimum value is -2 at $x = -1$ and $x = 3$ and point of inflection at $x = 1 \pm \frac{2\sqrt{3}}{3}$.</p> <p>11. Maximum value is $\frac{7}{2}$ at $x = -\frac{1}{2}$, minimum value is $-\frac{25}{2}$ at $x = \frac{3}{2}$, point of inflection at $x = \frac{1}{2}$</p> <p>12. (a) 18 (b) max. value = $\frac{4a^3}{27}$ at $(\frac{2a}{3}, \frac{a}{3})$</p> <p>13. $l = 9, b = 9$</p> <p>14. 225 sq.m</p> <p>15. 16. 600 m^2</p> | <p>(b) $x = 2, 3$</p> <p>(ii) Increasing
(ii) Increasing</p> <p>(b) $x = \frac{1}{2}$</p> <p>(c) does not exist</p> | <p>(c) $\pm 1, -2$</p> <p>(iii) decreasing
(iii) decreasing</p> |
|---|---|--|

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1.11 Rate Measures

In our practical life the components are interrelated (or interconnected). That is, the change in one of the components affects to the another component. Such as while driving a vehicle, as the force increases the acceleration will be increased. Similarly, if the price of an article (or commodity) is increased, then its demand will be decreased. Likewise the heavy rainfall possibly causes the land-slides and harmful floats and so on. Thus in the eco-system of surroundings, the change of one of the components causes the change of another component. The change of independent component causes the change of another component. The change of independent component changes the dependent component. In the above examples, the force is the independent variable whereas the acceleration of the vehicle is dependent variable. Similarly the price of a commodity is independent and that of its demand is the dependent variable. Thus in practice of mathematics, the **rate measure** is invented as a tool of measurement of the rate of change of dependent variable with respect to the rate of change of independent variable. Thus the rate measure can be defined as the rate measure is a tool which measures the rate of change of dependent variable (say y) with respect to the rate of change of independent variable (say x) denoted by $\frac{dy}{dx}$. In the earlier chapter, we had discussed that $\frac{dy}{dx}$ measures the slope of tangent to a curve as

well as it measures the velocity of a moving particle on a straight line. But now we can apply it (i.e. $\frac{dy}{dx}$) to measure the rate of change of components and hence called the rate measure. Here, we are going to deal with some examples involving this view point.

Note:

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|----------------|---|--|
| 1. Circle: | (i) $A = \pi r^2$ | (ii) $c = 2\pi r$ |
| 2. Sphere: | (i) $V = \frac{4}{3} \pi r^3$ | (ii) $s = 4\pi r^2$ |
| 3. Cylinder: | (i) $V = \pi r^2 h$ | (ii) $s = 2\pi r(r + h)$ |
| 4. Cone: | (i) $V = \frac{1}{3} \pi r^2 h$ | (ii) $s = \pi r(l + r)$ where l = slant height |
| 5. Cube: | (i) $V = l^3$ | (ii) $S = 6l^2$ |
| 6. Cosine law: | $a^2 = b^2 + c^2 - 2bc \cos A$ and so on. | |
- where, V = volume, A = Area, s = surface area
 c = circumference of circle, r = radius
 l = side, h = height

Example: 1

If the side of a cube is increasing at the rate of 0.5 cm/sec, find the rate of increasing of its (a) surface area when the side is 3 cm (b) volume when the side is 4 cm.

Solution:

Let x = side of the cube, s = surface area and V = volume.

Given that $\frac{dx}{dt} = 0.5$ cm/sec

- (a) Here, $x = 3$ cm (given), $\frac{ds}{dt} = ?$

We have, $s = 6x^2$

Differentiating with respect to time t we have,

$$\frac{ds}{dt} = 12 \times \frac{dx}{dt} = 12 \times 3 \times 0.5 = 18$$

- $\therefore$ The rate of increase in surface area is $\frac{ds}{dt} = 18$ sq-cm/sec.

- (b) Here, $\sin(x) = 4$ cm and $\frac{dV}{dt} = ?$

We have, $V = x^3$

Differentiating with respect to time t , it gives

$$\begin{aligned}\frac{dV}{dt} &= 3x^2 \frac{dx}{dt} = 3 \times (4)^2 \times 0.5 \\ &= 3 \times 16 \times 0.5 \\ &= 48 \times 0.5 = 24 \text{ cm}^3\end{aligned}$$

- $\therefore$ The rate of increase in volume is $\frac{dV}{dt} = 24$ cm³/sec.

Example: 2

Gasoline is pumped into a vertical cylindrical tank at a rate of 32 cu.ft./min. The radius of the tank is 4 feet. How fast is the surface area rising [CTEVT 2061/066 R/B]

Solution:

Let r = radius of the tank

h = height of the water level and

V = volume of water in the tank in time t

Given that $\frac{dV}{dt} = 32$ cu.ft/min, $r = 4$ ft.

Here, $V = \pi r^2 h$

Now, $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} = \pi(4)^2 \frac{dh}{dt} = 16\pi \frac{dh}{dt}$

or, $32 = 16\pi \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{2}{\pi}$ ft/min.

Example: 3

A spherical ball of salt dissolving in water decrease its volume at the rate of 0.75 cm³/min. Find the rate at which the radius of the ball is decreasing when the radius is 6 cm.

[CTEVT 2071 R/B]

Solution:

Let V = volume of the spherical ball of salt in time t

r = radius of the ball

Here, given that $\frac{dV}{dt} = -0.75 \text{ cm}^3/\text{min}$ [- sign in the fact that the volume is decreasing]

$$\frac{dr}{dt} = ? \text{ and } r = 6 \text{ cm}$$

Here, $V = \frac{4}{3} \pi r^3$

$$\begin{aligned} \text{Now, } \frac{dV}{dt} &= \frac{4}{3} \pi \times 3r^2 \frac{dr}{dt} \\ &= 4\pi r^2 \frac{dr}{dt} \end{aligned}$$

$$\begin{aligned} \text{or, } \frac{dV}{dt} &= 4\pi (6)^2 \frac{dr}{dt} = 4 \times 36 \pi \frac{dr}{dt} \\ &= 144 \pi \frac{dr}{dt} \end{aligned}$$

$$\text{or, } -0.75 = 144 \pi \frac{dr}{dt}$$

$$\text{or, } \frac{dr}{dt} = \frac{-0.75}{144\pi} = -\frac{75}{14400\pi} = -\frac{1}{192\pi}$$

$\therefore$ The rate of decrease in radius i.e., $\frac{dr}{dt} = \frac{1}{192\pi} \text{ cm/min.}$

Example: 4

A man of 6 ft. height walks away from 12 ft. lamp post at the rate of 4 ft./sec. How fast is his shadow lengthening when he is 21 ft from the post?

Solution:

In figure, AB = lamp post of height = 12 ft.

CD = man of height = 6 ft.

D be position of man in time t and E be the end of shadow.

Let $BD = x$ and $DE = y$ then $BE = x + y$. Since $\triangle ABE$ and $\triangle CDE$ are similar. So we have

$$\frac{AB}{CD} = \frac{BE}{DE}$$

$$\text{or, } \frac{12}{6} = \frac{x+y}{y}$$

$$\text{or, } 2 = \frac{x+y}{y}$$

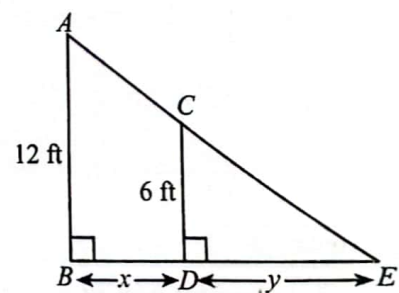
$$\text{or, } 2y = x + y$$

$$\text{or, } y = x$$

$$\text{Now, } \frac{dy}{dt} = \frac{dx}{dt}$$

$$\text{When } x = 21, \frac{dx}{dt} = 4 \text{ ft./sec}$$

$$\therefore \frac{dy}{dt} = \frac{dx}{dt} = 4 \text{ ft./sec.}$$



Example: 5

If the area of a circle increases at a uniform rate prove that the rate of increase of the perimeter varies inversely as the radius.
[CTEVT 2065 R/B, 2066 R]

Solution:

Let, r = radius of the circle in time t
 c = circumference of the circle in the t and
 A = area of the circle.

Here, given that, $\frac{dA}{dt} = k$, (i.e. uniform rate)

$$\text{We have, } A = \pi r^2 \quad (1)$$

$$c = 2\pi r \quad (2)$$

From (1),

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

$$\text{or, } k = 2\pi r \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = \frac{k}{2\pi r} \quad (3)$$

Again, from (2)

$$\frac{dc}{dt} = 2\pi \frac{dr}{dt}$$

$$\text{or, } \frac{dc}{dt} = 2\pi \times \frac{k}{2\pi r} \quad [\because \text{Using (3)}]$$

$$\text{or, } \frac{dc}{dt} = \frac{k}{r}$$

$$\text{or, } \frac{dc}{dt} = kr^{-1}$$

This shows that the rate of increase of perimeter $\frac{dc}{dt}$ varies inversely as the radius.

Example: 6

Water flows into an inverted conical tank at the rate of $24 \text{ ft}^3/\text{min}$. When the depth of water is 3 ft, how fast is the level rising? Assume that the height of the tank is 6 ft. and the radius at the top is 1 ft.

Solution:

Let ABC be the conical tank where, AD = height of the tank = 6 ft.

AE = height of water level in time t = 3 ft.

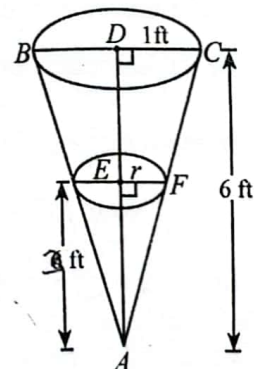
DC = radius of the tank at the top = 1 ft.

Also, let h = height of water level = AE

r = radius of water surface at time t = EF .

and V = volume of water in time t

Here, given that $\frac{dV}{dt} = 24 \text{ cu.ft./min}$.



$$\text{and } \frac{dh}{dt} = ?$$

In figure, since AEF and ADC are similar triangles.

$$\text{So, we have, } \frac{AE}{AD} = \frac{EF}{DC}$$

$$\text{or, } \frac{h}{6} = \frac{r}{1}$$

$$\text{or, } h = 6r$$

$$\therefore r = \frac{1}{6}h = \frac{h}{6}$$

$$\text{Here, we have } V = \frac{1}{3} \pi r^2 h$$

$$\text{or, } V = \frac{1}{3} \pi \left(\frac{h}{6}\right)^2 \cdot h$$

$$\text{or, } V = \frac{1}{3} \pi \frac{h^3}{36} = \frac{\pi h^3}{3 \times 36}$$

$$\text{Now, } \frac{dV}{dt} = \frac{3\pi h^2}{3 \times 36} \frac{dh}{dt} = \frac{\pi}{36} (3)^2 \quad [\because h = 3 \text{ given}]$$

$$\text{or, } 24 = \frac{9}{36} \pi \cdot \frac{dh}{dt} = \frac{\pi dh}{4 dt} \quad \left[\because \frac{dV}{dt} = 24 \text{ given} \right]$$

$$\text{or, } \frac{dh}{dt} = \frac{4 \times 24}{\pi} = \frac{96}{\pi}$$

$$\therefore \frac{dh}{dt} = \frac{96}{\pi} \text{ ft./min}$$

Example: 7

An aeroplane is flying above a town horizontally at a height of $\frac{2}{3}$ miles with velocity 15 miles/hr. Find the rate at which it is receding from a fixed point on the ground which it passed over two minutes ago. [CTEVT 2065 R]

Solution:

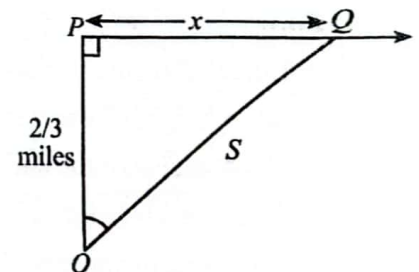
Let O be a fixed point, OP be the height of plane which is flying horizontally above O (a town) from P to Q i.e., Q be the position of the aeroplane at time t . Also let $PQ = x$, $OQ = S$. Here $OP = \frac{2}{3}$ miles and $\frac{dx}{dt} = 15$ miles/hr.

Now, $PQ = x = \text{distance covered by the plane in 2 minutes.}$

$$\text{or, } x = \frac{15}{60} \times 2 = \frac{2}{4} = \frac{1}{2} \text{ miles.}$$

In $\triangle OPQ$,

$$(OQ)^2 = (OP)^2 + (PQ)^2$$



$$\text{or, } S^2 = \left(\frac{2}{3}\right)^2 + x^2 \quad (1)$$

$$\text{or, } S^2 = \frac{4}{9} + \left(\frac{1}{2}\right)^2 \quad [\because x = 1/2]$$

$$\text{or, } S^2 = \frac{4}{9} + \frac{1}{4} = \frac{16 + 9}{36} = \frac{25}{36} = \left(\frac{5}{6}\right)^2$$

$$\therefore S = \frac{5}{6} \text{ miles}$$

Differentiating (1) with respect to t . It gives,

$$2S \frac{dS}{dt} = 0 + 2x \frac{dx}{dt}$$

$$\text{or, } S \frac{dS}{dt} = x \frac{dx}{dt}$$

$$\text{or, } \frac{5}{6} \frac{dS}{dt} = \frac{1}{2} \times 15$$

$$\text{or, } \frac{dS}{dt} = \frac{15}{2} \times \frac{6}{5} = 9$$

$$\therefore \frac{dS}{dt} = 9 \text{ miles/hr.}$$

$$\left[\begin{array}{l} \because 15 \text{ miles/hr. means} \\ 60 \text{ minutes} = 15 \text{ miles} \\ 1 \text{ minute} = 15/60 \text{ miles} \\ 2 \text{ minutes} = \frac{15 \times 2}{60} = \frac{1}{2} \text{ miles} \end{array} \right]$$

EXERCISE 1(IV)

Difficulty Level: A

1. The radius of a circle is increasing at the rate of 1.5 cm/sec. Find the rate of increase of its (a) area when the radius is 2 cm (b) circumference.
2. If the side of a square is increasing at the rate of 5 m/sec find the rate of increase of its (a) area when its side is 3 m (b) perimeter.
3. The radius of a circle is increasing uniformly at the rate of 0.3 cm/sec. At what rate is the area increasing when the radius is 10 cm?
4. A spherical balloon is inflated at the rate of 10 cubic inches/sec. At what rate is the radius is increasing when its radius is 1 inch?

Difficulty Level: B

5. A spherical ball of salt is dissolving in water in such a manner that the rate of decrease in volume at any instant is proportional to the surface. Prove that the radius is decreasing at a constant rate.
6. Water is poured into a right circular cylinder of radius 2 ft. at the rate of 20 cu.ft./min. Find the rate at which the level of water is rising in the cylinder.
7. Water is pumped into a right circular cylinder of radius 8 cm at the rate of 18 cm³/min. Find the rate at which the level of water is rising in the cylinder.
8. If dry sand spills to form a conical pile with height equal to its radius of the base, how fast is the height increasing when the height is 3 ft. and the sand is being added at the rate 2 cu. ft/min?
9. If the volume of an expanding cube is increasing at the rate of 4 cu.ft/min., how fast is its surface area increasing when the surface area is 24 sq.ft.?

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10. If the volume of an expanding cube is increasing at the rate of $24 \text{ cm}^3/\text{min}$, how fast is the surface area increasing when the surface area is 216 cm^2 ?
11. Two aeroplanes in flight cross above a town at 1:00 p.m. one plane travels east at 300 miles/hr, the other north at 400 miles/hr. At what rate does the distance between the planes change at 3:00 p.m.?
12. Water flows into an inverted conical tank at the rate of 27 cu.ft/min . When the depth of water is 2 fts, how fast is the level of water rising? Assume that the height of the tank is 4 fts and the radius at the top is 2 ft.
13. A kite is 24 meters high and there are 25 meters of cord out. If the kite moves horizontally at the rate of 36 km/hr directly away from the person who is flying it, how fast is the cord paid out?
14. Two cars start at the same time from the junction of two roads one on each road, with uniform speed of V miles/hr. If the roads are inclined at 120° , find the rate of increase of distance between the two cars.

ANSWERS

- | | |
|-------------------------------------|--|
| 1. (a) $6\pi \text{ sq. cm/sec}$ | (b) $3\pi \text{ cm/sec}$ |
| 2. (a) 30 sq. m/sec | (b) 20 m/sec |
| 3. $6\pi \text{ cm}^2/\text{sec}$ | 4. $\frac{5}{2\pi} \text{ inches/sec}$ |
| 6. $\frac{5}{\pi} \text{ ft/min}$ | 7. $\frac{9}{32\pi} \text{ cm/min.}$ |
| 8. $\frac{2}{9\pi} \text{ ft/min}$ | 9. 8 sq.ft/min |
| 10. 16 sq.cm/min | 11. 500 miles/hr |
| 12. $\frac{27}{\pi} \text{ ft/min}$ | 13. 10.08 km/hr |
| 14. $\sqrt{3} V \text{ miles/hr}$ | |

□□□

1.12 Indeterminate Forms

In calculus, we face determinate and indeterminate forms in calculating limits. A mathematical expression, having a single number or infinity while evaluating limit, is called a determinate form. For example,

$$\lim_{x \rightarrow 1} \frac{2x+3}{x+1} = \frac{2 \times 1 + 3}{1+1} = \frac{2+3}{2} = \frac{5}{2}$$

A mathematical expression, that does not possess a single (number) value while operating the limit is called the indeterminate form. Moreover, an indeterminate form is a mathematical expression involving at most 2 of 0, 1 or ∞ after evaluating the limit. In other words, on the process of evaluating the limits of a function sometimes it may have the results which can not possess a fixed number (i.e. a definite number). In this case, the results are said to be in indeterminate form. We discuss about the different indeterminate form with examples as below:

There are seven indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \times \infty$, 0^0 , 1^∞ and ∞^0 . But here we discuss only two indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ according to the syllabus. For examples:

$$1. \quad \lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = \frac{1^3 - 1}{1 - 1} = \frac{1 - 1}{1 - 1} = \frac{0}{0}$$

$$2. \quad \lim_{x \rightarrow \infty} (\sqrt{x} - \sqrt{x+2}) = \sqrt{\infty} - \sqrt{\infty+2} = \sqrt{\infty} - \sqrt{\infty} = (\infty - \infty)$$

$$3. \quad \lim_{x \rightarrow \infty} \frac{x-a}{\sqrt{x^2-a^2}} = \frac{\infty-a}{\sqrt{\infty^2-a^2}} = \frac{\infty}{\sqrt{\infty^2-a^2}} = \frac{\infty}{\sqrt{\infty}} = \frac{\infty}{\infty} \text{ etc.}$$

Note:

The expression $\infty + \infty = \infty$, $-\infty - \infty = -\infty$, $0^0 = 0$, $0^{-\infty} = \infty$, $\infty \times \infty = \infty$ are the determinate forms.

1.13 Evaluation of the Indeterminate Forms

(a) Factorization method

If the limit takes form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ while giving the limit, first it can be evaluated by factorization (or simplifying) then we use the given limit to find the required limiting value.

Here let us see an example,
$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$$

$$= \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{(x-3)} = 3 + 3 = 6$$

(b) Rationalization Method

Consider the example,

$$\begin{aligned}
 & \lim_{x \rightarrow a} \frac{\sqrt{x+a} - \sqrt{3x-a}}{x-a} \quad \left(\frac{0}{0} \text{ form} \right) \\
 = & \lim_{x \rightarrow a} \frac{(\sqrt{x+a} - \sqrt{3x-a})(\sqrt{x+a} + \sqrt{3x-a})}{(x-a)(\sqrt{x+a} + \sqrt{3x-a})} \\
 = & \lim_{x \rightarrow a} \frac{x+a-3x+a}{(x-a)(\sqrt{x+a} + \sqrt{3x-a})} \\
 = & \lim_{x \rightarrow a} \frac{-2(x-a)}{(x-a)(\sqrt{x+a} + \sqrt{3x-a})} \\
 = & \lim_{x \rightarrow a} \frac{-2}{(\sqrt{x+a} + \sqrt{3x-a})} \\
 = & \frac{-2}{\sqrt{2a} + \sqrt{2a}} = \frac{-2}{2\sqrt{2a}} = -\frac{1}{\sqrt{2a}}
 \end{aligned}$$

(c) Using the formula

For all rational values of n ,

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

For example, we have

$$\begin{aligned}
 \lim_{x \rightarrow a} \frac{x^{2/3} - a^{2/3}}{x - a} &= \frac{2}{3} \cdot a^{2/3-1} = \frac{2}{3} a^{-1/3} \\
 &= \frac{2}{3a^{1/3}}
 \end{aligned}$$

L' Hospital's Theorem

If $f(x)$ and $g(x)$ are continuous functions with their derivatives $f'(x)$ and $g'(x)$ at $x = a$ such that $f(a) = g(a) = 0$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}, \text{ provided that } g'(a) \neq 0.$$

Note:

If $f'(a) = g'(a) = 0$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f''(x)}{g''(x)} = \frac{f''(a)}{g''(a)}, \text{ and so on.}$$

That is, When the limit takes $\frac{0}{0}$ form, then differentiate Numerator and denominator separately with respect to the given variable there in, and put the limit. If again takes $\frac{0}{0}$ form, then repeat the process.

Example: 1

Evaluate $\lim_{x \rightarrow 3} \frac{x^3 - 27}{x - 3}$

Solution:

It takes $\frac{0}{0}$ form when $x \rightarrow 3$. Using L' Hospital's Rule, we get

$$\begin{aligned} & \lim_{x \rightarrow 3} \frac{3x^2 - 0}{1 - 0} \\ &= \frac{3 \times 3^2}{1} = 3 \times 9 = 27 \end{aligned}$$

Note: We can evaluate this problem by factorization method and by using formula also.

Example: 2

Evaluate $\lim_{x \rightarrow 0} \frac{3 - \sqrt{x^2 + 9}}{x^2}$

Solution:

It takes $\frac{0}{0}$ form when $x \rightarrow 0$. We have,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{3 - \sqrt{x^2 + 9}}{x^2} &= \lim_{x \rightarrow 0} \frac{3 - \sqrt{x^2 + 9}}{x^2} \times \frac{3 + \sqrt{x^2 + 9}}{3 + \sqrt{x^2 + 9}} \\ &= \lim_{x \rightarrow 0} \frac{9 - x^2 - 9}{x^2 (3 + \sqrt{x^2 + 9})} \\ &= \lim_{x \rightarrow 0} \frac{-x^2}{x^2 (3 + \sqrt{x^2 + 9})} = \lim_{x \rightarrow 0} \frac{-1}{3 + \sqrt{x^2 + 9}} \\ &= -\frac{1}{3 + \sqrt{0 + 9}} = -\frac{1}{3 + 3} = -\frac{1}{6} \end{aligned}$$

Example: 3

Evaluate $\lim_{x \rightarrow \infty} \frac{4x^2 - 5x}{3 - 2x^2}$

Solution: We have

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{4x^2 - 5x}{3 - 2x^2} \quad \left(\frac{\infty}{\infty} \text{ form} \right) \\ &= \lim_{x \rightarrow \infty} \frac{4 - \frac{5}{x}}{\frac{3}{x^2} - 2} \quad (\text{dividing by } x^2) \\ &= \frac{4 - \frac{5}{\infty}}{\frac{3}{\infty^2} - 2} = \frac{4 - 0}{0 - 2} = -\frac{4}{2} = -2 \end{aligned}$$

Example: 4

Evaluate: $\lim_{x \rightarrow 1} \frac{1 + \log x - x}{1 - 2x + x^2}$

Solution:

Here, $\lim_{x \rightarrow 1} \frac{1 + \log x - x}{1 - 2x + x^2}$

$\left[\frac{0}{0} \text{ form as } x \rightarrow 1 \right]$

$$= \lim_{x \rightarrow 1} \frac{0 + \frac{1}{x} - 1}{0 - 2 + 2x} = \lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{-2 + 2x}$$

$\left[\frac{0}{0} \text{ form} \right]$

$$= \lim_{x \rightarrow 1} \frac{-\frac{1}{x^2} - 0}{0 + 2} = \frac{-1}{2} = -\frac{1}{2} \times \frac{1}{2} = -\frac{1}{2}$$

Example: 5

Evaluate $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$

Solution: It takes $\frac{\infty}{\infty}$ form. So we have,

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$$

$$= \lim_{x \rightarrow \infty} \frac{e^x}{2x}$$

(L' Hospital's Rule)

$$= \lim_{x \rightarrow \infty} \frac{e^x}{2}$$

$$= \frac{e^\infty}{2} = \infty$$

Example: 6

Evaluate $\lim_{x \rightarrow 0} \frac{x \cos x - \log(1+x)}{x^2}$

Solution: We have

$$\lim_{x \rightarrow 0} \frac{x \cos x - \log(1+x)}{x^2}$$

$\left(\frac{0}{0} \text{ form} \right)$

$$= \lim_{x \rightarrow 0} \frac{\cos x - x \sin x - \frac{1}{1+x}}{2x}$$

(L' Hospital's Rule)

$$= \lim_{x \rightarrow 0} \frac{-\sin x - \sin x - x \cos x + \frac{1}{(1+x)^2}}{2} = \frac{0 - 0 + \frac{1}{(1+0)^2}}{2}$$

$$= \frac{1}{2}$$

Example: 7

Evaluate $\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{x - 1}$

Solution: We have

$$\begin{aligned}
 & \lim_{x \rightarrow 1} \frac{x^{\frac{1}{3}} - 1}{x - 1} && \left(\frac{0}{0} \text{ form}\right) \\
 = & \lim_{x \rightarrow 1} \frac{\frac{1}{3}x^{-\frac{2}{3}} - 0}{1 - 0} && [\because \text{From L' Hospital's Rule}] \\
 = & \frac{1}{3} \lim_{x \rightarrow 1} \frac{1}{x^{\frac{2}{3}}} \\
 = & \frac{1}{3} \frac{1}{\frac{2}{3}} = \frac{1}{3}
 \end{aligned}$$

Example: 8

Compute $\lim_{x \rightarrow \pi} \frac{x^2 - \pi^2}{\sin x}$

Solution: We have

$$\begin{aligned}
 & \lim_{x \rightarrow \pi} \frac{x^2 - \pi^2}{\sin x} && \left(\frac{0}{0} \text{ form}\right) \\
 = & \lim_{x \rightarrow \pi} \frac{2x - 0}{\cos x} \\
 = & \frac{2\pi}{\cos \pi} \\
 = & \frac{2\pi}{-1} = -2\pi
 \end{aligned}$$

Example: 9

Evaluate $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - \sqrt{x^2 - 1})$

Solution: We have

$$\begin{aligned}
 & \lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - \sqrt{x^2 - 1}) && [(\infty - \infty) \text{ form}] \\
 = & \lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - \sqrt{x^2 - 1}) \times \frac{(\sqrt{x^2 + 1} + \sqrt{x^2 - 1})}{(\sqrt{x^2 + 1} + \sqrt{x^2 - 1})} \\
 = & \lim_{x \rightarrow \infty} \frac{x^2 + 1 - x^2 + 1}{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}} \\
 = & \lim_{x \rightarrow \infty} \left(\frac{2}{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}} \right) = \frac{2}{\infty + \infty} = \frac{2}{\infty} = 0.
 \end{aligned}$$

Example: 10

Evaluate $\lim_{x \rightarrow 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right)$

Solution: It takes $\infty - \infty$ form.

We have,

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right) \\
 = & \lim_{x \rightarrow 0} \frac{(\sin^2 x - x^2)}{x^2 \sin^2 x} \quad \left(\frac{0}{0} \text{ form} \right) \\
 = & \lim_{x \rightarrow 0} \frac{2 \sin x \cos x - 2x}{2x \sin^2 x + 2x^2 \sin x \cos x} \quad [\because \text{Using L' Hospital's Rule}] \\
 = & \lim_{x \rightarrow 0} \frac{\sin 2x - 2x}{2x \sin^2 x + x^2 \sin 2x} \quad \left(\frac{0}{0} \text{ form} \right) \\
 = & \lim_{x \rightarrow 0} \frac{2 \cos 2x - 2}{2 \sin^2 x + 4x \sin x \cos x + 2x \sin 2x + x^2 2 \cos 2x} \\
 = & \lim_{x \rightarrow 0} \frac{2 \cos 2x - 2}{2 \sin^2 x + 2x \sin 2x + 2x \sin 2x + 2x^2 \cos 2x} \\
 = & \lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\sin^2 x + 2x \sin 2x + x^2 \cos 2x} \quad \left(\frac{0}{0} \text{ form} \right) \\
 = & \lim_{x \rightarrow 0} \frac{-2 \sin 2x}{2 \sin x \cos x + 2 \sin 2x + 4x \cos 2x + 2x \cos 2x - 2x^2 \sin 2x} \\
 & \quad [\because \text{From L' Hospital's Rule}] \\
 = & \lim_{x \rightarrow 0} \frac{-2 \sin 2x}{3 \sin 2x + 6x \cos 2x - 2x^2 \sin 2x} \\
 = & \frac{-4 \cos 2x}{6 \cos 2x + 6 \cos 2x - 12x \sin 2x - 4x \sin 2x - 4x^2 \cos 2x} \\
 = & \frac{-4 \cos 0}{6 \cos 0 + 6 \cos 0 - 0 - 0 - 0} = \frac{-4}{12} = -\frac{1}{3}.
 \end{aligned}$$

EXERCISES I (V)**Difficulty Level: A****1. Evaluate the limits**

(a) $\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4}$

(b) $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

(c) $\lim_{t \rightarrow 1} \frac{5t^4 - 4t^2 - 1}{10 - t - 9t^3}$

(d) $\lim_{x \rightarrow 0} \frac{\tan x - x}{x - \sin x}$

(e) $\lim_{x \rightarrow \infty} \frac{x^n}{e^x}$

(g) $\lim_{x \rightarrow 0} \frac{e^x - x - 1}{x^2}$

(f) $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 5x}{\tan x}$

(h) $\lim_{x \rightarrow 1} \frac{x^3 - 2x^2 + 2x - 4}{x^2 - 5x + 6}$

Difficulty Level: B**2. Evaluate the following Limits**

(a) $\lim_{x \rightarrow 2} \frac{x^3 - 2x^2 + 2x - 4}{x^2 - 5x + 6}$

(c) $\lim_{t \rightarrow e} \frac{\log t - 1}{t - e}$

(e) $\lim_{x \rightarrow 0} \frac{\log(1 - x^2)}{\log \cos x}$

(g) $\lim_{x \rightarrow a} (a - x) \tan\left(\frac{\pi x}{2a}\right)$

(i) $\lim_{x \rightarrow 0} \frac{x^2 - \sin^2 x}{x^2}$

(b) $\lim_{x \rightarrow \pi} \frac{\cos \frac{x}{2}}{\pi - x}$

(d) $\lim_{x \rightarrow 0} \frac{xe^x - \log(1 + x)}{x^2}$

(f) $\lim_{x \rightarrow 0} \frac{\cos hx - \cos x}{x \sin x}$

(h) $\lim_{x \rightarrow 0} \frac{x - \sin^{-1} x}{\sin^3 x}$

(j) $\lim_{x \rightarrow 0} \frac{x - \sin x \cos x}{x^3}$

3. Evaluate

(a) $\lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$

(c) $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 5x}{\tan x}$

(e) $\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$

(b) $\lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{x^2} \log(1 + x) \right)$

(d) $\lim_{x \rightarrow 0} \sin x \log x^2$

(f) $\lim_{x \rightarrow 0} \frac{\tan x - x}{x^2 \tan x}$

ANSWERS

- | | | | | |
|----------------------|----------------------|--------------------|-------------------|-------------------------|
| 1. (a) 8 | (b) 1 | (c) $\frac{-3}{7}$ | (d) 2 | (e) 0 |
| (f) $\frac{1}{5}$ | (g) $\frac{1}{2}$ | (h) -6 | | |
| 2. (a) -6 | (b) $\frac{1}{2}$ | (c) $\frac{1}{e}$ | (d) $\frac{3}{2}$ | (e) 2 |
| (f) 1 | (g) $\frac{2a}{\pi}$ | (h) $-\frac{1}{6}$ | (i) 0 | (j) $\frac{2}{3}$ |
| 3. (a) $\frac{1}{2}$ | (b) $\frac{1}{2}$ | (c) $\frac{1}{5}$ | (d) 0 | (e) 2 (f) $\frac{1}{3}$ |



Partial Derivatives

Unit

2

2.1. Introduction

The notation of continuity, limit and differentiation in relation to real valued functions of two real variables, will be briefly explained in this chapter. A few theorems of elementary character will also be given. The subject of functions of two variable is capable of extension to functions of any number n of variables, but the treatment of the subject in this generalized form is not within the scope of this book. Only a few examples dealing with functions of two or three variables are given.

2.2. Functions of Two or More Variables

If three variables z, x, y are so related such that for every pair of values of x and y within the defined domain, z has a single definite value then z is said to be a function of the two independent variables x and y . In this case, we write $z = f(x, y)$.

More generally, if the three variables z, x, y are so related that z is determined when x and y are known, z is said to be a function of the two independent variables x and y .

Since the area of a triangle is determined when its base and altitude are given, area of a triangle is a function of its base and altitude. Similarly, the volume of a gas is a function of its pressure and temperature. Also, the volume of a parallelepiped is a function of three variables: length, breadth and height.

2.3. Limits and Continuity

A function of two variables $z = f(x, y)$ is said to tend to a limit L as (x, y) tends to (a, b) , if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all (x, y) in the domain of f .

That is, $|f(x, y) - L| < \epsilon$ whenever $|(x, y) - (a, b)| < \delta$.

In such a case we write

$$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = L.$$

In a similar way, a limit of a function of three or more independent variables can be defined.

The function $z = f(x, y)$ is said to be continuous at the point (a, b) , if corresponding to a pre-assigned positive number ϵ , however small, there exists a positive number δ , such that

$$|f(x, y) - f(a, b)| < \epsilon$$

whenever $0 \leq |x - a| \leq \delta$ and $0 \leq |y - b| \leq \delta$.

In other words, a function $z = f(x, y)$ is said to be continuous at a point (a, b) if and only if

$$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = f(a, b).$$

In a similar way, continuity of a function of three or more variables can be defined.

2.4. Partial Derivatives

The result of differentiating $z = f(x, y)$ with respect to x , treating y as a constant, is called the partial derivative of z with respect to x , denoted by one of the symbols $\frac{\partial z}{\partial x}$, $\frac{\partial f}{\partial x}$, $f_x(x, y)$, f_x , z_x etc. The

partial derivative of the function of two variables is given by $f_x(x, y) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$ provided this limit exists.

Similarly, the partial derivatives of $z = f(x, y)$ with respect to y is the function of two variables given by

$$f_y(x, y) = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \text{ provided this limit exists.}$$

If $w = f(x, y, z)$, then the partial derivative of w with respect to x is the function of three variables given by

$$f_x(x, y, z) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}.$$

Similarly, we have

$$f_y(x, y, z) = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

$$\text{and } f_z(x, y, z) = \lim_{\Delta z \rightarrow 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

Note: The del ∂ is generally used to denote the partial derivative in order to distinguish it from the symbol d of ordinary derivative.

Example: 1

Use definition of partial derivatives to find $f_x(x, y)$ and $f_y(x, y)$ for the function $f(x, y) = x^2y^2$.

Solution:

We have $f(x, y) = x^2y^2$

By definition, we have

$$\begin{aligned} f_x(x, y) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 y^2 - x^2 y^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{y^2 [x^2 + 2x\Delta x + (\Delta x)^2 - x^2]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{y^2 [2x\Delta x + (\Delta x)^2]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{y^2 \Delta x (2x + \Delta x)}{\Delta x} \end{aligned}$$

$$\begin{aligned}
&= \lim_{\Delta x \rightarrow 0} y^2 (2x + \Delta x) \\
&= y^2 (2x + 0) \\
&= 2xy^2
\end{aligned}$$

Similarly for partial derivative with respect to y , we have

$$\begin{aligned}
f_y(x, y) &= \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \\
&= \lim_{\Delta y \rightarrow 0} \frac{x^2 (y + \Delta y)^2 - x^2 y^2}{\Delta y} \\
&= \lim_{\Delta y \rightarrow 0} \frac{x^2 [y^2 + 2y\Delta y + (\Delta y)^2 - y^2]}{\Delta y} \\
&= \lim_{\Delta y \rightarrow 0} \frac{x^2 [2y\Delta y + (\Delta y)^2]}{\Delta y} \\
&= \lim_{\Delta y \rightarrow 0} x^2 (2y + \Delta y) \\
&= x^2 (2y + 0) \\
&= 2x^2 y
\end{aligned}$$

Example: 2

If $z = 3x^3y^3 - 9x^2y + xy^2 + 4y$, find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial z}{\partial x} \Big|_{(1,0)}$ and $\frac{\partial z}{\partial y} \Big|_{(1,0)}$.

Solution:

We have $z = 3x^3y^3 - 9x^2y + xy^2 + 4y$

Differentiating both sides partially with respect to x gives

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 3(3x^2)y^3 - 9(2x)y + (1)y^2 + 0 \\
&= 9x^2y^3 - 18xy + y^2
\end{aligned}$$

Differentiating both sides partially with respect to y gives

$$\begin{aligned}
\frac{\partial z}{\partial y} &= 3x^3(3y^2) - 9x^2(1) + x(2y) + 4(1) \\
&= 9x^3y^2 - 9x^2 + 2xy + 4
\end{aligned}$$

$$\text{Thus, } \frac{\partial z}{\partial x} \Big|_{(1,0)} = 9(1)^2(0)^3 - 18(1)(0) + 0^2 = 0$$

$$\frac{\partial z}{\partial y} \Big|_{(1,0)} = 9(1)^3(0)^2 - 9(1)^2 + 2(1)(0) + 4 = -5$$

Example: 3

If $f(x, y) = xy^2 + x^2y$, find $f_x(x, y)$ and $f_y(x, y)$. Also find $f_x(3, 4)$ and $f_y(3, 4)$.

Solution:

We have, $f(x, y) = xy^2 + x^2y$

To find $f_x(x, y)$, we treat y as a constant and differentiate f with respect to x :

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x}(xy^2 + x^2y) \\ &= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial x}(x^2y) \\ &= y^2 \frac{\partial x}{\partial x} + y \frac{\partial x^2}{\partial x} \\ &= y^2 \cdot 1 + y \cdot 2x \\ &= y^2 + 2xy \end{aligned}$$

To find $f_y(x, y)$ we treat x as a constant and differentiate f with respect to y :

$$\begin{aligned} f_y(x, y) &= \frac{\partial}{\partial y}(xy^2 + x^2y) \\ &= \frac{\partial}{\partial y}(xy^2) + \frac{\partial}{\partial y}(x^2y) \\ &= x \frac{\partial y^2}{\partial y} + x^2 \frac{\partial y}{\partial y} \\ &= x \cdot 2y + x^2 \cdot 1 \\ &= 2xy + x^2 \end{aligned}$$

Also, we have $f_x(3, 4) = 4^2 + 2(3)(4) = 40$

and $f_y(3, 4) = 2(3)(4) + 3^2 = 33$.

Example: 4

Show that $\lim_{x \rightarrow 0} \left(\lim_{y \rightarrow 0} \frac{x-y}{x+y} \right) \neq \lim_{y \rightarrow 0} \left(\lim_{x \rightarrow 0} \frac{x-y}{x+y} \right)$.

Solution:

$$\begin{aligned} \text{L.H.S.} &= \lim_{x \rightarrow 0} \left(\lim_{y \rightarrow 0} \frac{x-y}{x+y} \right) \\ &= \lim_{x \rightarrow 0} \frac{x}{x} \\ &= \lim_{x \rightarrow 0} 1 \\ &= 1 \\ \text{R.H.S.} &= \lim_{y \rightarrow 0} \left(\lim_{x \rightarrow 0} \frac{x-y}{x+y} \right) \end{aligned}$$

$$\begin{aligned}
 &= \lim_{y \rightarrow 0} \frac{-y}{y} \\
 &= \lim_{y \rightarrow 0} (-1) \\
 &= -1
 \end{aligned}$$

$$\text{Hence, } \lim_{x \rightarrow 0} \left(\lim_{y \rightarrow 0} \frac{x-y}{x+y} \right) \neq \lim_{y \rightarrow 0} \left(\lim_{x \rightarrow 0} \frac{x-y}{x+y} \right).$$

2.5. Successive Partial Derivatives

Since each of the partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ is in general a function of x and y , each may possess partial derivatives with respect to these two independent variables, and these are called the second order partial derivatives of $z = f(x, y)$. The usual notations for these second order partial derivatives are

$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right)$	i.e. $\frac{\partial^2 z}{\partial x^2}$	or	f_{xx} etc.
$\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right)$	i.e. $\frac{\partial^2 z}{\partial y^2}$	or	f_{yy} etc.
$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right)$	i.e. $\frac{\partial^2 z}{\partial x \partial y}$	or	f_{xy}
$\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right)$	i.e. $\frac{\partial^2 z}{\partial y \partial x}$	or	f_{yx}

Notes:

1. If $z = f(x, y)$, then partial derivatives of z are very often denoted by the following notations:

$$\begin{aligned}
 \frac{\partial z}{\partial x} &= p, & \frac{\partial z}{\partial y} &= q, & \frac{\partial^2 z}{\partial x^2} &= r \\
 \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial^2 z}{\partial y \partial x} = s, & \frac{\partial^2 z}{\partial y^2} &= t
 \end{aligned}$$

2. For most of the functions that occur in applications we have

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.$$

That is, the partial derivative has the same value whether we differentiate partially first with respect to x and then with respect to y or the reverse.

Example: 5

Calculate all the first order and second order partial derivatives of $f(x, y) = e^{-x} \sin(x + y)$.

Solution:

$$\text{We have } f(x, y) = e^{-x} \cdot \sin(x + y)$$

$$\begin{aligned}
 \text{Now, } f_x(x, y) &= \frac{\partial}{\partial x} [e^{-x} \sin(x+y)] \\
 &= e^{-x} \frac{\partial}{\partial x} \sin(x+y) + \sin(x+y) \frac{\partial e^{-x}}{\partial x} \\
 &= e^{-x} \cos(x+y) \frac{\partial}{\partial x} (x+y) + \sin(x+y) \cdot e^{-x} \frac{\partial}{\partial x} (-x) \\
 &= e^{-x} \cos(x+y) \cdot 1 + \sin(x+y) e^{-x} (-1) \\
 &= e^{-x} [\cos(x+y) - \sin(x+y)]
 \end{aligned}$$

$$\begin{aligned}
 f_y(x, y) &= \frac{\partial}{\partial y} [e^{-x} \sin(x+y)] \\
 &= e^{-x} \frac{\partial}{\partial y} \sin(x+y) \\
 &= e^{-x} \cos(x+y) \frac{\partial}{\partial y} (x+y) \\
 &= e^{-x} \cos(x+y) \cdot 1 \\
 &= e^{-x} \cos(x+y)
 \end{aligned}$$

$$\begin{aligned}
 f_{xx}(x, y) &= \frac{\partial}{\partial x} e^{-x} [\cos(x+y) - \sin(x+y)] \\
 &= e^{-x} \frac{\partial}{\partial x} [\cos(x+y) - \sin(x+y)] + [\cos(x+y) - \sin(x+y)] \frac{\partial e^{-x}}{\partial x} \\
 &= e^{-x} [-\sin(x+y) \cdot 1 - \cos(x+y) \cdot 1] + [\cos(x+y) - \sin(x+y)] e^{-x} (-1) \\
 &= e^{-x} [-\sin(x+y) - \cos(x+y) - \cos(x+y) + \sin(x+y)] \\
 &= e^{-x} [-2 \cos(x+y)] \\
 &= -2e^{-x} \cos(x+y)
 \end{aligned}$$

$$\begin{aligned}
 f_{yy}(x, y) &= \frac{\partial}{\partial y} [e^{-x} \cos(x+y)] \\
 &= e^{-x} \frac{\partial}{\partial y} \cos(x+y) \\
 &= e^{-x} [-\sin(x+y) \cdot 1] \\
 &= -e^{-x} \sin(x+y)
 \end{aligned}$$

$$\begin{aligned}
 f_{xy}(x, y) &= \frac{\partial}{\partial y} e^{-x} [\cos(x+y) - \sin(x+y)] \\
 &= e^{-x} \frac{\partial}{\partial y} [\cos(x+y) - \sin(x+y)] \\
 &= e^{-x} [\sin(x+y) \cdot 1 - \cos(x+y) \cdot 1] \\
 &= -e^{-x} [\sin(x+y) + \cos(x+y)]
 \end{aligned}$$

$$\begin{aligned}
 f_{yx}(x, y) &= \frac{\partial}{\partial x} [e^{-x} \cos(x+y)] \\
 &= e^{-x} \frac{\partial}{\partial x} \cos(x+y) + \cos(x+y) \frac{\partial e^{-x}}{\partial x} \\
 &= e^{-x} [-\sin(x+y) \cdot 1] + \cos(x+y) \cdot e^{-x} (-1) \\
 &= -e^{-x} [\sin(x+y) + \cos(x+y)]
 \end{aligned}$$

Example: 6

If $u = \sqrt{x^2 + y^2 + z^2}$, then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{u}$.

Solution:

We have $u = \sqrt{x^2 + y^2 + z^2}$

$$\text{Now, } \frac{\partial u}{\partial x} = \frac{1}{2} (x^2 + y^2 + z^2)^{1/2-1} \frac{\partial}{\partial x} (x^2 + y^2 + z^2)$$

$$= \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \times 2x$$

$$= \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{x}{u}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{x}{u} \right)$$

$$= \frac{u \frac{\partial x}{\partial x} - x \frac{\partial u}{\partial x}}{u^2}$$

$$= \frac{u - x \cdot \frac{x}{u}}{u^2}$$

$$= \frac{u^2 - x^2}{u^3}$$

Similarly, we have

$$\frac{\partial^2 u}{\partial y^2} = \frac{u^2 - y^2}{u^3}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{u^2 - z^2}{u^3}$$

$$\begin{aligned}
 \therefore \text{L.H.S.} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\
 &= \frac{u^2 - x^2}{u^3} + \frac{u^2 - y^2}{u^3} + \frac{u^2 - z^2}{u^3}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{u^2 - x^2 + u^2 - y^2 + u^2 - z^2}{u^3} \\
&= \frac{3u^2 - (x^2 + y^2 + z^2)}{u^3} \\
&= \frac{3u^2 - u^2}{u^3} \\
&= \frac{2u^2}{u^3} = \frac{2}{u} \\
&= \text{R.H.S.}
\end{aligned}$$

Example: 7

If $w = e^{xyz}$, show that $x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z} = 3wxyz$.

Solution:

We have $w = e^{xyz}$

$$\text{Now, } \frac{\partial w}{\partial x} = \frac{\partial}{\partial x} e^{xyz} = e^{xyz} \frac{\partial}{\partial x} (xyz) = e^{xyz} \cdot yz$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y} e^{xyz}$$

$$= e^{xyz} \frac{\partial}{\partial y} (xyz)$$

$$= e^{xyz} \cdot xz$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z} e^{xyz}$$

$$= e^{xyz} \frac{\partial}{\partial z} (xyz)$$

$$= e^{xyz} \cdot xy$$

$$\text{Thus, } x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z}$$

$$= x \cdot e^{xyz} \cdot yz + y \cdot e^{xyz} \cdot xz + z \cdot e^{xyz} \cdot xy$$

$$= 3e^{xyz} \cdot xyz$$

$$= 3wxyz$$

Example: 8

If $z = xe^{x-y} + ye^{y-x}$, show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = e^{x-y} + e^{y-x}$.

Solution:

We have $z = xe^{x-y} + ye^{y-x}$

$$\begin{aligned}
\text{Now, } \frac{\partial z}{\partial x} &= \frac{\partial}{\partial x} (xe^{x-y} + ye^{y-x}) \\
&= \frac{\partial}{\partial x} (xe^{x-y}) + y \frac{\partial}{\partial x} e^{y-x} \\
&= x \frac{\partial}{\partial x} e^{x-y} + e^{x-y} \frac{\partial x}{\partial x} + ye^{y-x} \frac{\partial}{\partial x} (y-x) \\
&= xe^{x-y} \frac{\partial}{\partial x} (x-y) + e^{x-y} + ye^{y-x} (-1) \\
&= xe^{x-y}(1) + e^{x-y} - ye^{y-x} \\
&= xe^{x-y} + e^{x-y} - ye^{y-x} \\
\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y} (xe^{x-y} + ye^{y-x}) \\
&= x \frac{\partial}{\partial y} (e^{x-y}) + \frac{\partial}{\partial y} (ye^{y-x}) \\
&= xe^{x-y} (-1) + y \frac{\partial e^{y-x}}{\partial y} + e^{y-x} \frac{\partial y}{\partial y} \\
&= -xe^{x-y} + ye^{y-x} (1) + e^{y-x} \\
&= -xe^{x-y} + ye^{y-x} + e^{y-x} \\
\text{Thus, } \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} &= xe^{x-y} + e^{x-y} - ye^{y-x} - xe^{x-y} + ye^{y-x} + e^{y-x} \\
&= e^{x-y} + e^{y-x}
\end{aligned}$$

Example: 9

A company manufactures two types of skis, the Lighting and the Alpine models. Suppose the joint cost function for producing x pairs of the Lighting model and y pairs of the Alpine model per week is

$$c = f(x, y) = 0.07x^2 + 75x + 85y + 6000$$

where c is expressed in dollars. Determine $\frac{\partial c}{\partial x}$ and $\frac{\partial c}{\partial y}$ where $x = 100$ and $y = 50$, and interpret the results.

Solution:

$$\text{We have } c = f(x, y) = 0.07x^2 + 75x + 85y + 6000$$

$$\begin{aligned}
\text{Now, } \frac{\partial c}{\partial x} &= \frac{\partial}{\partial x} (0.07x^2 + 75x + 85y + 6000) \\
&= 0.14x + 75
\end{aligned}$$

$$\begin{aligned}
\text{and } \frac{\partial c}{\partial y} &= \frac{\partial}{\partial y} (0.07x^2 + 75x + 85y + 6000) \\
&= 85
\end{aligned}$$

$$\text{Thus, } \left. \frac{\partial c}{\partial x} \right|_{(100, 50)} = 0.14(100) + 75 = 89 \quad (1)$$

$$\text{and } \left. \frac{\partial c}{\partial y} \right|_{(100, 50)} = 85 \quad (2)$$

Equation (1) means that increasing the output of the Lighting model from 100 to 101 while maintaining production of the Alpine model at 50 increases costs by approximately \$89.

Equation (2) means that increasing the output of the Alpine model from 50 to 51 and holding production of the Lighting model at 100 will increase costs by approximately \$85.

Example: 10

Find $\frac{\partial g}{\partial x}$ if $2x^2 + 3y^2 + 5z^2 = 900$.

Solution:

$$\text{We have } 2x^2 + 3y^2 + 5z^2 = 900$$

Differentiating both sides partially with respect to x gives

$$\frac{\partial}{\partial x} (2x^2 + 3y^2 + 5z^2) = \frac{\partial}{\partial x} (900)$$

$$\text{or, } \frac{\partial}{\partial x} (2x^2) + \frac{\partial}{\partial x} (3y^2) + \frac{\partial}{\partial x} (5z^2) = 0$$

$$\text{or, } 4x + 0 + 5.2z \frac{\partial z}{\partial x} = 0$$

$$\text{or, } 10z \frac{\partial z}{\partial x} = -4x$$

$$\text{or, } \frac{\partial z}{\partial x} = -\frac{2x}{5z}$$

EXERCISE - 2(I)

Difficulty Level: A

1. Find the partial derivative of the function with respect to each of the variables.

(a) $f(x, y) = 2x^2 + 3xy + 4y^2 + 5x + 6y - 7$

(b) $f(x, y) = \sqrt[3]{x^2 + y^2}$

(c) $g(x, y, z) = 2xy^2z^6 - 4x^2y^3z^2 + 3xyz$

Difficulty Level: B

2. Use definition method to find the partial derivatives of the followings:

(a) $f(x, y) = x^2y - xy^2$

(b) $f(x, y) = xy^2 - x^2$

(c) $f(x, y) = xy + y^2$

(d) $f(x, y) = x^2 - xy$

3. If $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$, prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$.

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4. If $u = \sin^{-1} \frac{x}{y} + \tan^{-1} \frac{y}{x}$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$.
5. Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ if
- (a) $u = \log(x^2 + y^2)$ (b) $u = \tan^{-1} \frac{y}{x}$
- (c) $u = e^x (x \cos y - y \sin y)$
6. Find $f_x(x, y)$ and $f_y(x, y)$ if $f(x, y) = \frac{5y}{y + \sin x}$.
7. Find f_{xx}, f_{yy}, f_{yx} and f_{xy} for the following functions
- (a) $f(x, y) = x^3 + 3x^2y + 3xy^2 + y^3$ (b) $f(x, y) = x \cos y + y \cos x$
8. If $f(x, y, z) = z^2(3x^2 - 4xy^3)$, find $f_{xyz}(1, 2, 3)$.
9. For $f(x, y) = e^{xy}$, show that $f_{xx}(x, y) + f_{xy}(x, y) + f_{yx}(x, y) + f_{yy}(x, y) = f(x, y)[(x + y)^2 + 2]$.
10. Find the value of $\frac{\partial^3 x}{\partial z \partial y \partial x} \Big|_{(1,2,3)}$ if $w = (2x + 3y + 4z)^3$.

ANSWERS

1. (a) $f_x(x, y) = 4x + 3y + 5$ (b) $f_x(x, y) = \frac{2x}{3(x^2 + y^2)^{2/3}}$
- $f_y(x, y) = 3x + 8y + 6$ $f_y(x, y) = \frac{2y}{3(x^2 + y^2)^{2/3}}$
- (c) $g_x = 2y^2z^2 - 8xy^3z^2 + 3yz$ $g_y = 4xyz^6 - 12x^2y^2z^2 + 3xz$
- $g_z = 12xy^2z^5 - 8x^2y^3z + 3xy$
2. (a) $f_x(x, y) = 2xy - y^2$ (b) $f_x(x, y) = y^2 - 2x$
- $f_y(x, y) = x^2 - 2xy$ $f_y(x, y) = 2xy$
- (c) $f_x(x, y) = y$ (d) $f_x(x, y) = 2x - y$
- $f_y(x, y) = x + 2y$ $f_y(x, y) = -x$
6. $f_x(x, y) = -\frac{5y \cos x}{(y + \sin x)^2}$ $f_y(x, y) = \frac{5 \sin x}{(y + \sin x)^2}$
7. (a) $f_{xx} = f_{xy} = f_{yx} = f_{yy} = 6(x + y)$ $f_{xy} = f_{yx} = -(\sin x + \sin y)$
- (b) $f_{xx} = -y \cos x$ $f_{yy} = -x \cos y$
8. -288 10. 144

□□□

2.6. Homogeneous Functions and Euler's Theorem

A function $f(x, y)$ is said to be homogeneous of degree n in variables x and y if it can be expressed in the form as

$$x^n \phi\left(\frac{y}{x}\right) \text{ or } y^n \phi\left(\frac{x}{y}\right).$$

If V is a homogeneous function of degree n in x, y, z , then each of $\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}$ is a homogeneous function of degree $n - 1$.

$$\text{Since } ax^2 + 2hxy + by^2 = x^2 \left\{ a + 2h\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 \right\} = x^2 \phi\left(\frac{y}{x}\right),$$

$ax^2 + 2hxy + by^2$ is a homogeneous function of degree 2 in x and y .

Similarly, $\frac{y}{x}, x \tan^{-1}\left(\frac{y}{x}\right), x^2 \log\left(\frac{y}{x}\right)$ are homogeneous functions of degree 0, 1, 2 respectively.

An alternative test for a function $f(x, y)$ to be homogeneous of degree n is that $f(tx, ty) = t^n f(x, y)$ for all values of t , where t is independent of x and y .

Similarly, a function $f(x, y, z)$ is said to be homogeneous of degree n in the variables x, y, z if it can be put in the form

$$x^n f\left(\frac{y}{x}, \frac{z}{x}\right) \text{ or } f(tx, ty, tz) = t^n f(x, y, z); \text{ and so on, for any number of variables.}$$

Example: 1

If $u = f(x, y) = 2x^3 + 5x^2y + xy^2$ then show that it is a homogeneous function of degree 3.

Solution:

$$\begin{aligned} \text{Here, } f(x, y) &= 2x^3 + 5x^2y + xy^2 \\ &= x^3 \left[2 + 5\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 \right] \\ &= x^3 \left[2\left(\frac{y}{x}\right)^0 + 5\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 \right] \\ &= x^3 \phi\left(\frac{y}{x}\right) \end{aligned}$$

$\therefore u = f(x, y)$ is a homogeneous function of degree 3.

Alternative method,

$$\text{We have, } u = f(x, y) = 2x^3 + 5x^2y + xy^2$$

$$\begin{aligned} \text{or, } f(tx, ty) &= 2(tx)^3 + 5(tx)^2(ty) + (tx)(ty)^2 \\ &= t^3[2x^3 + 5x^2y + xy^2] \\ &= t^3 f(x, y) \end{aligned}$$

$\therefore u = f(x, y)$ is a homogeneous function of degree 3.

Euler's Theorem of Homogenous Functions

If $f(x, y)$ be a homogenous functions of x and y of degree n , then $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf(x, y)$.

Proof:

Since $f(x, y)$ is a homogeneous function of degree n , let

$$f(x, y) = x^n \phi\left(\frac{y}{x}\right) = x^n \phi(V) \text{ where } V = \frac{y}{x}.$$

$$\begin{aligned} \text{Now, } \frac{\partial f}{\partial x} &= nx^{n-1} \phi(V) + x^n \phi'(V) \frac{\partial V}{\partial x} \\ &= nx^{n-1} \phi(V) + x^n \phi'(V) \left(-\frac{y}{x^2}\right) \end{aligned}$$

$$\therefore x \frac{\partial f}{\partial x} = nx^n \phi(V) - x^{n-1} y \phi'(V) \quad (1)$$

$$\text{Again, } \frac{\partial f}{\partial y} = x^n \phi'(V) \frac{\partial V}{\partial y} = x^n \phi'(V) \cdot \frac{1}{x}$$

$$\therefore y \frac{\partial f}{\partial y} = x^{n-1} y \phi'(V) \quad (2)$$

Adding (1) and (2) we get,

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nx^n \phi(V) - x^{n-1} y \phi'(V) + x^{n-1} y \phi'(V)$$

$$\text{or, } x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nx^n \phi(V) = nf(x, y)$$

$$\therefore x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf(x, y)$$

Note: If $f(x, y, z)$ is a homogeneous function of degree n , then $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = nf(x, y, z)$.

Example: 2

Show that $u = f(x, y) = ax^2 + 2hxy + by^2$ is homogeneous function and verify Euler's theorem.

Solution:

$$\text{We have } u = f(x, y) = ax^2 + 2hxy + by^2$$

$$\begin{aligned} \text{Now, } f(tx, ty) &= a(tx)^2 + 2h(tx)(ty) + b(ty)^2 \\ &= t^2(ax^2 + 2hxy + by^2) \\ &= t^2 f(x, y). \end{aligned}$$

This shows that $u = f(x, y)$ is a homogeneous function of degree 2. For the verification of Euler's theorem, we have

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} (ax^2 + 2hxy + by^2)$$

$$= 2ax + 2hy$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} (ax^2 + 2hxy + by^2)$$

$$= 2hx + 2by$$

$$\begin{aligned} \therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} &= x(2ax + 2hy) + y(2hx + 2by) \\ &= 2ax^2 + 2hxy + 2hxy + 2by^2 \\ &= 2(ax^2 + 2hxy + by^2) \\ &= 2u \end{aligned}$$

Thus, Euler's theorem is verified.

Example: 3

If $u = \tan^{-1} \frac{x^3 + y^3}{x - y}$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$

Solution:

$$\text{We have } u = \tan^{-1} \frac{x^3 + y^3}{x - y}$$

$$\begin{aligned} \text{or, } \tan u &= \frac{x^3 + y^3}{x - y} \\ &= \frac{x^3 \left\{ 1 + \left(\frac{y}{x} \right)^2 \right\}}{x \left\{ 1 - \left(\frac{y}{x} \right) \right\}} \\ &= x^2 \phi \left(\frac{y}{x} \right) \end{aligned}$$

Thus, $\tan u$ is a homogeneous function of degree 2.

Let $\tan u = v$. Then, by Euler's theorem, we have

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = 2v$$

$$\text{or, } x \frac{\partial(\tan u)}{\partial x} + y \frac{\partial(\tan u)}{\partial y} = 2 \tan u$$

$$\text{or, } x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \tan u$$

$$\begin{aligned} \text{or, } x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} &= \frac{2 \tan u}{\sec^2 u} \\ &= 2 \sin u \cos u = \sin 2u \end{aligned}$$

2.7. Derivative of Composite Functions (Total Differential)

We have already defined the differential of a function of a single independent variable. We now give the corresponding definition of the differential of a function of two independent variables x and y . Thus if

$$u = f(x, y) \text{ we define } du = f_x dx + f_y dy.$$

$$\text{Also, if } u = f(x, y, z), \text{ then } du = f_x dx + f_y dy + f_z dz$$

Example: 4

Find $\frac{dz}{dt}$ when $z = e^{x+y}$, $x = t^2 + 3$ and $y = \sqrt{t}$.

Solution:

$$\text{We have } z = e^{x+y}$$

$$x = t^2 + 3$$

$$y = \sqrt{t}$$

$$\begin{aligned} \text{Now, } \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= \frac{\partial}{\partial x} (e^{x+y}) \cdot \frac{d}{dt} (t^2 + 3) + \frac{\partial}{\partial y} (e^{x+y}) \cdot \frac{d}{dt} (t^{1/2}) \\ &= e^{x+y} \cdot 2t + e^{x+y} \cdot \frac{3}{2} t^{1/2} \\ &= \left(2t + \frac{3}{2} \sqrt{t} \right) e^{x+y} \end{aligned}$$

Example: 5

Find $\frac{du}{dt}$ when $u = x^2 + y^2 + z^2$, $x = 2t + 1$, $y = t + 5$ and $z = 7t$.

Solution:

$$\text{We have, } u = x^2 + y^2 + z^2$$

$$x = 2t + 1$$

$$y = t + 5$$

$$z = 7t$$

$$\begin{aligned} \text{Now, } \frac{du}{dt} &= \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt} \\ &= (2x)(2) + (2y)(1) + (2z)(7) \\ &= 4x + 2y + 14z \\ &= 4(2t + 1) + 2(t + 5) + 14(7t) \\ &= 8t + 4 + 2t + 10 + 98t = 108t + 14 \end{aligned}$$

2.8. Chain rule

Let $z = f(x, y)$, where both x and y are functions of r and s given by

$$x = x(r, s)$$

and $y = y(r, s)$.

If f, x and y have continuous partial derivatives, then z is a function of r and s and

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

The chain rule can be extended. For example, suppose $z = f(u, v, w)$ and u, v, w are all functions of r, s and t . Then

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial r} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial r} + \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial r}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial s} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial s} + \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial s}$$

$$\text{and } \frac{\partial z}{\partial t} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial t} + \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial t}$$

Now, consider the situation where $z = f(x, y)$ such that $x = x(t)$ and $y = y(t)$. Then z is a function of single variable t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

If $y = f(x)$ and $x = x(r, s)$, then

$$\frac{\partial y}{\partial r} = \frac{dy}{dx} \cdot \frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial y}{\partial s} = \frac{dy}{dx} \cdot \frac{\partial x}{\partial s}.$$

Example: 6

If $z = 5x + 3y$, $x = 2r + 3s$ and $y = r - 2s$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$.

Solution:

We have $z = 5x + 3y$

$$x = 2r + 3s$$

$$y = r - 2s$$

$$\text{Now, } \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= \frac{\partial}{\partial x} (5x + 3y) \cdot \frac{\partial}{\partial r} (2r + 3s) + \frac{\partial}{\partial y} (5x + 3y) \cdot \frac{\partial}{\partial r} (r - 2s)$$

$$= 5 \times 2 + 3 \times 1$$

$$= 10 + 3$$

$$= 13$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= \frac{\partial}{\partial x} (5x + 3y) \cdot \frac{\partial}{\partial s} (2r + 3s) + \frac{\partial}{\partial y} (5x + 3y) \cdot \frac{\partial}{\partial s} (r - 2s)$$

$$= 5 \times 3 + 3(-2)$$

$$= 15 - 6$$

$$= 9$$

Example: 7

If $w = f(x, y, z) = 3x^2y + xyz - 4x^2z^3$, $x = 2r - 3s$, $y = 6r + s$ and $z = r - s$, find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$.

Solution:

We have, $w = f(x, y, z) = 3x^2y + xyz - 4y^2z^3$

$$x = 2r - 3s$$

$$y = 6r + s$$

$$z = r - s$$

$$\text{Now, } \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$= (6xy + yz)(2) + (3x^2 + xz - 8yz^3)(6) + (xy - 12y^2z^2)(1)$$

$$= x(18x + 13y + 6z) + 2yz(1 - 24z^2 - 6yz)$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$

$$= (6xy + yz)(-3) + (3x^2 + xz - 8yz^3)(1) + (xy - 12y^2z^2)(-1)$$

$$= x(3x - 19y + z) - yz(3 + 8z^2 - 12yz)$$

Example: 8

Find $\frac{\partial y}{\partial r}$ when $y = x^2 - 7x + 5$ and $x = 19rs + 2s^2r^2$.

Solution:

We have, $y = x^2 - 7x + 5$

$$x = 19rs + 2s^2r^2$$

$$\text{Now, } \frac{\partial y}{\partial r} = \frac{dy}{dx} \cdot \frac{\partial x}{\partial r}$$

$$= (2x - 7) \cdot (19s) = 19s(2x - 7)$$

Example: 9

Find $\frac{du}{dt}$ of $u = e^{xyz}$, $x = t^3$, $y = \frac{1}{t}$, $z = e^t$.

Solution:

Here, $u = e^{xyz}$

$$x = t^3, y = \frac{1}{t}, z = e^t$$

$$\begin{aligned} \text{We have, } \frac{du}{dt} &= \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt} \\ &= e^{xyz} \cdot yz \cdot 3t^2 + e^{xyz} \cdot zx \cdot \left(-\frac{1}{t^2}\right) + e^{xyz} \cdot xy \cdot e^t \end{aligned}$$

$$\begin{aligned} \text{or, } \frac{du}{dt} &= e^{xyz} \left[yz \cdot 3t^2 + zx \left(-\frac{1}{t^2}\right) + xy \cdot e^t \right] \\ &= e^{xyz} \left[\frac{1}{t} \cdot e^t \cdot 3t^2 + e^t \cdot t^3 \left(-\frac{1}{t^2}\right) + t^3 \cdot \left(\frac{1}{t}\right) \cdot e^t \right] \\ &= e^{xyz} [3t e^t - t e^t + t^2 e^t] \\ &= e^{t^3} \cdot \left(\frac{1}{t}\right) \cdot e^t [t e^t - t e^t + t^2 e^t] \\ &= e^{t^3} \cdot e^t e^t [3t + t + t^2] \\ &= e^{t^3} \cdot e^t e^t [t^2 + 2t] \\ &= e^{(t^3+t)} t [t+2] \\ &= e^{t(t^2+1)} t [t+2] \end{aligned}$$

Example: 10

If $u = x^3 + y^3$, $x = \cos t$ and $y = \sin t$, find $\frac{du}{dt}$.

Solution:

We have $u = x^3 + y^3$

$$x = \cos t$$

$$y = \sin t$$

$$\begin{aligned} \text{Now, } \frac{du}{dt} &= \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \\ &= (3x^2) (-\sin t) + (3y^2) (\cos t) \\ &= 3(-x^2 \sin t + y^2 \cos t) \\ &= 3(-\cos^2 t \sin t + \sin^2 t \cos t) \\ &= 3 \sin t \cos t (\sin t - \cos t) \end{aligned}$$

Example: 11

If $z = (4x - 3y)^3$, where $x = r^2s$ and $y = r - 2s$, evaluate $\frac{\partial z}{\partial r}$ when $r = 0$ and $s = 1$.

Solution:

We have $z = (4x + 3y)^3$

$$x = r^2s$$

$$y = r - 2s$$

$$\begin{aligned}\text{Now, } \frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r} \\ &= 3(4x + 3y)^2 \cdot 4 \cdot 2rs + 3(4x + 3y)^2 \cdot 3 \cdot 1 \\ &= 24rs(4x + 3y)^2 + 9(4x + 3y)^2 \\ &= (24rs + 9)(4x + 3y)^2\end{aligned}$$

Given that $r = 0$ and $s = 1$, So,

$$x = 0 \text{ and } y = -2$$

$$\begin{aligned}\therefore \left. \frac{\partial z}{\partial r} \right|_{\substack{r=0 \\ s=1}} &= (0 + 9)(0 - 6)^2 \\ &= 9 \times 36 = 324\end{aligned}$$

EXERCISE - 2(II)**Difficulty Level: B**

1. Show that the following functions are homogeneous and verify Euler's theorem.

(a) $f(x, y) = x^3 + y^3$

(b) $f(x, y, z) = x^3 + y^3 + z^3 - 3xyz$

(c) $u = \frac{x^2 + y^2}{xy + yz}$

(d) $u = \sin^{-1} \left(\frac{x^2 + y^2}{x + y} \right)$

2. If $u = \cos^{-1} \left(\frac{x + y}{\sqrt{x} + \sqrt{y}} \right)$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + \frac{1}{2} \cot u = 0$.

3. If $u = \sin^{-1} \left(\frac{x^2 + y^2}{x + y} \right)$, prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \tan u$.

4. Check $f(x, y) = x^n \cdot \tan \left(\frac{y}{x} \right)$ for homogeneity and verify Euler's theorem if homogeneous.

5. Using total derivative, find $\frac{du}{dt}$ if $u = x^2 + y^2$, $x = at^2$ and $y = 2at$.

6. Find $\frac{du}{dt}$ of $u = x^2 + y^2$, $x = \cos t + \sin t$, $y = \cos t - \sin t$ at $t = 0$.

7. Use the chain rule to find the derivative of $w = x^2y - y^2$, where $x = \sin t$ and $y = e^t$. Also evaluate $\frac{dw}{dt}$ when $t = 0$.

8. Find $\frac{dw}{dt}\bigg|_{t=0}$ if $w = xy + z$, $x = \cos t$, $y = \sin t$ and $z = t$.
9. Find $\frac{du}{dt}$ if
- (a) $u = e^{xyz}$, $x = t^3$, $y = \frac{1}{t}$ and $z = e^t$
- (b) $u = \ln(x^2 + y^2 + z^2)$, $x = 2 - 3t$, $y = t^2 + 3$ and $z = 4 - t$.
10. Given that $z = e^{xy}$, $x = r - 4s$ and $y = r - s$, find $\frac{\partial z}{\partial r}$ in terms of r and s .
11. Use chain rule to evaluate $\frac{\partial z}{\partial s}$ when $r = -2$, $s = 5$ and $t = 4$ where $z = \frac{x + e^y}{y}$, $x = rs + se^{rt}$, $y = 9 + rt$.
12. Use the idea of total differential to find the approximate change in the volume of the cylinder of radius 5 cm and height 10 cm if we decrease the radius by 0.01 cm and increases the height by 0.02 cm.

ANSWERS

5. $\frac{du}{dt} = 4a^2t(t^2 + 2)$
6. $\frac{du}{dt}\bigg|_{t=0} = 0$
7. $\frac{dw}{dt} = 2xy \cos t + (x^2 - 2y)e^t$
- $\frac{dw}{dt}\bigg|_{t=0} = -2$
8. $\frac{dw}{dt}\bigg|_{t=0} = 2$
9. (a) $xyz e^{xyz} \left(1 + \frac{2}{t}\right)$
- (b) $\frac{-6x + 4yt - 2z}{x^2 + y^2 + z^2}$
10. $(2r - 5s) e^{r^2 - 5rs + 4s^2}$
11. $-2 + e^{-8}$
12. Volume decreases by $0.5\pi \text{ cm}^3$



Applications of Anti-derivatives

3.1 Introduction

Ant-derivative is used to find the area of the plane region which is under a curve and of the plane regions between two curves. Particularly, the area of the plane region under a curve can be obtained by the method of limit of sum. But the definite integral is used to find the area of the plane regions between two curves as well as the region under a curve both.

To compute the area of the plane regions such as, circles, triangles, quadrilaterals and the other such plane regions whose sides are straight lines (or which are enclosed by straight lines), there we already had some special methods. But if one of the sides is in the form of a curve, for that there were no such special methods to find the area of such plane regions. A great mathematician named Leibnitz introduced a method called 'Limit of sum' in 1676 A.D. to find the area of such plane regions. In this method a plane region under a curve is divided into n infinitely tiny strips (i.e. infinitely small n equal parts) such that each part nearly forms a rectangle. Then the process forms a series of the area of the rectangles and thus the sum of the series gives the area of the plane region.

Before studying of the application of anti-derivatives here we first study the standard integrals and its related problems so that we may need to take the idea of rational fractions. Thus at first we discuss about the integration of rational fractions after that we go to the standard integrals.

3.2 Integration of Rational Fractions

The integration of the algebraic expressions which are in quotient form of two polynomials may sometimes be difficult. To make their integration easier, the rational fractions are to be expressed into more than one partial fractions. In the first case, if the degree of the numerator is greater than that of the denominator then we first divide the numerator by the denominator to express the improper rational fraction as the sum of a polynomial and a proper fraction. Then we decompose the proper fraction into partial fractions. In the second case, if the given rational fraction is itself a proper fraction, we express the proper rational fraction into the partial fractions. We have some simple rules of breaking the algebraic rational fractions into the partial fractions which are as follows:

- | | | |
|-----|---|--|
| (a) | If $f(x) = \frac{q(x)}{(ax+b)^n}$ | then $f(x) = \frac{A}{(ax+b)} + \frac{B}{(ax+b)^2} + \dots + \frac{N}{(ax+b)^n}$ |
| (b) | If $f(x) = \frac{q(x)}{(ax+b)(cx+d)}$ | then $f(x) = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$ |
| (c) | If $f(x) = \frac{q(x)}{(ax+b)(cx^2+d)}$ | then $f(x) = \frac{A}{(ax+b)} + \frac{Bx+c}{(cx^2+d)}$ |

- (d) If $f(x) = \frac{q(x)}{(ax+b)(cx+d)^2}$ then $f(x) = \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$
 where $A, B, C, \dots, N$ are the constants to be determined.

Example: 1

Integrate the following breaking into the partial fractions:

(a) $\int \frac{13}{12x^2 + 19x + 4} dx$

(b) $\int \frac{(2x+1)}{(x-1)(x^2+1)} dx$

Solution:

(a) Here, $I = \int \frac{13}{12x^2 + 19x + 4} dx$
 $= \int \frac{13}{12x^2 + 16x + 3x + 4} dx$

or, $I = \int \frac{13}{(3x+4)(4x+1)} dx$

Let, $\frac{13}{(3x+4)(4x+1)} = \frac{A}{3x+4} + \frac{B}{4x+1}$ (1)

Clearly, we have

$$13 = A(4x+1) + B(3x+4)$$

If $x = -\frac{1}{4}$, then $13 = A(0) + B\left[3 \times \left(-\frac{1}{4}\right) + 4\right] = 0 + B\left(\frac{-3+16}{4}\right)$

$$\Rightarrow 13 = \frac{13}{4} B \quad \therefore B = 4$$

Again, if $x = -\frac{4}{3}$ then $13 = A\left(4 \times \frac{4}{3} + 1\right) + B(0)$

$$\Rightarrow 13 = A\left(-\frac{16+3}{3}\right) = -\frac{13}{3} A$$

$$\therefore A = -3$$

For the values of A and B the relation (1) takes the form as,

$$\frac{13}{(3x+4)(4x+1)} = \frac{-3}{3x+4} + \frac{4}{4x+1}$$

Integrating both sides with respect to 'x', we get

$$\int \frac{13}{(3x+4)(4x+1)} dx = -3 \int \frac{1}{3x+4} dx + 4 \int \frac{1}{4x+1} dx$$

or, $\int \frac{13}{(3x+4)(4x+1)} dx = \frac{-3 \log(3x+4)}{3} + \frac{4 \log(4x+1)}{4} + c$

$$\therefore I = \int \frac{13}{12x^2 + 19x + 4} dx = \log\left(\frac{4x+1}{3x+4}\right) + c$$

(b) Here, $I = \frac{(2x+1)}{(x-1)(x^2+1)} dx$

Let $\frac{(2x+1)}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$ (1)

or, $2x+1 = A(x^2+1) + (Bx+C)(x-1)$

Equating the constants term, $A - C = 1$ (2)

Equating the coefficients of x , $C - B = 2$ (3)

And equating the coefficient of x^2 , $A + B = 0$

$\Rightarrow A = -B$ (4)

Adding (2) and (3),

$A - B = 3$ (5)

Using (4) in (5) we get,

$-B - B = 3$

$\Rightarrow -2B = 3 \Rightarrow B = -\frac{3}{2}$

From (4), $A = -\left(-\frac{3}{2}\right) = \frac{3}{2}$

From (2), $\frac{3}{2} - C = 1 \Rightarrow C = \frac{3}{2} - 1 = \frac{1}{2}$

$\therefore A = \frac{3}{2}, B = -\frac{3}{2}$ and $C = \frac{1}{2}$

Now, equation (1) becomes as

$$\frac{(2x+1)}{(x-1)(x^2+1)} = \frac{3/2}{x-1} + \frac{-3/2x+1/2}{x^2+1}$$

Integrating we get,

$$I = \int \frac{(2x+1)}{(x-1)(x^2+1)} dx = \frac{3}{2} \int \frac{1}{x-1} dx - \frac{1}{2} \int \frac{3x-1}{x^2+1} dx$$

or, $I = \int \frac{(2x+1)}{(x-1)(x^2+1)} dx = \frac{3}{2} \log(x-1) - \frac{1}{2} \cdot \frac{3}{2} \int \frac{2x-2/3}{x^2+1} dx$

or, $I = \frac{3}{2} \log(x-1) - \frac{3}{4} \log(x^2+1) + \frac{1}{2} \tan^{-1} x + c$

$\therefore I = \frac{3}{2} \log(x-1) - \frac{3}{4} \log(x^2+1) + \frac{1}{2} \tan^{-1} x + c$

EXERCISE - 3 (I)**Difficulty Level: A**

1. Evaluate the following using the limit of sum (summation method):

(a) $\int_0^1 x^2 dx$

(b) $\int_0^1 (2x^2 - 3) dx$

(c) $\int_0^b (2x + 1) dx$

(d) $\int_0^1 e^x dx$

Difficulty Level: B

2. Evaluate (using partial fraction):

(a) $\int \frac{2x}{(2x+3)(3x+5)} dx$

(b) $\int \frac{3x}{(x-a)(x-b)} dx$

(c) $\int \frac{2x^2 + 3}{x^3 + 3x^2 + 2x} dx$

(d) $\int \frac{(x+1)dx}{(x+2)(x^2+1)}$

ANSWERS

1. (a) $\frac{1}{3}$ (b) $\frac{3a^3}{3} - 3a$ (c) $b(1+b)$ (d) $(e-1)$

2. (a) $-3 \log(2x+3) + \frac{10}{3} \log(3x+5) + c$
 (b) $\frac{3}{(a-b)} [a \log(x-a) - b \log(x-b)] + c$
 (c) $\frac{3}{2} \log x - 5 \log(x+1) + \frac{11}{2} \log(x+2) + c$
 (d) $-\frac{1}{5} \log(x+2) + \frac{1}{10} \log(x^2+1) + \frac{3}{5} \tan^{-1}(x) + c$

□□□

3.3 Standard Integrals

Upto now we had integrated the given integrand by using the general rules of integration. Such as algebraic substitution, trigonometric substitution, integration by parts etc. But sometimes the given integrand may not be integrals only by using the above mentioned methods. For that we need some other special fundamental techniques. Such integrals are known as the standard integrals. The standard integrals have been divided into different groups according as the different techniques to be used during the process of integration.

Standard Integrals I

In this section, we discuss the integrals which are directly related to the differentiation formulae and are considered as the fundamental integrals. These are:

$$\begin{aligned}
 \text{(a)} \quad \int \frac{1}{a^2 + x^2} dx &= \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c & \text{(b)} \quad \int \frac{1}{\sqrt{a^2 + x^2}} dx &= \log (x + \sqrt{a^2 + x^2}) + c \\
 \text{(c)} \quad \int \frac{1}{\sqrt{a^2 - x^2}} dx &= \sin^{-1} \left(\frac{x}{a} \right) + c & \text{(d)} \quad \int \frac{1}{\sqrt{x^2 - a^2}} dx &= \log (x + \sqrt{x^2 - a^2}) + c \\
 \text{(e)} \quad \int \frac{1}{a^2 - x^2} dx &= \frac{1}{2a} \log \left(\frac{a+x}{a-x} \right) + c & \text{(f)} \quad \int \frac{1}{x^2 - a^2} dx &= \frac{1}{2a} \log \frac{x-a}{x+a} + c.
 \end{aligned}$$

Proof:

$$\text{(a)} \quad \text{Here, } I = \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

$$\text{L.H.S. (} = I \text{)} = \int \frac{1}{a^2 + x^2} dx$$

$$\text{Let } x = a \tan \theta \Rightarrow \theta = \tan^{-1} \left(\frac{x}{a} \right)$$

$$\text{Differentiating we get, } dx = a \sec^2 \theta d\theta$$

$$\text{Now, } I = \int \frac{1}{a^2 + a^2 \tan^2 \theta} \cdot a \sec^2 \theta d\theta = \frac{1}{a} \int \frac{\sec^2 \theta d\theta}{\sec^2 \theta}$$

$$\text{or, } I = \frac{1}{a} \int d\theta = \frac{1}{a} \theta + c$$

$$\therefore I = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

Similar procedure can be used to prove (b), (c) and (d).

$$\text{(e)} \quad \text{Here, } I = \int \frac{1}{a^2 - x^2} dx = \int \frac{1}{(a+x)(a-x)} dx$$

$$= \frac{1}{2a} \int \left[\frac{1}{a+x} + \frac{1}{a-x} \right] dx$$

[$\because$ Using the partial fraction]

$$\text{or, } I = \frac{1}{2a} [\log (a+x) - \log (a-x)] + c$$

$$\text{or, } I = \frac{1}{2a} \log \left(\frac{a+x}{a-x} \right) + c$$

$$\therefore \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left(\frac{a+x}{a-x} \right) + c$$

Similar procedure is used to prove (f).

Integrals reducible to the form of any one of the above standard integrals I

The integrals:

$$(a) \int \frac{1}{ax^2 + bx + c} dx$$

$$(b) \int \frac{1}{\sqrt{ax^2 + bx + c}} dx$$

$$(c) \int \frac{sx + e}{ax^2 + bx + c} dx$$

$$(d) \int \frac{sx + e}{\sqrt{ax^2 + bx + c}} dx$$

can be evaluated on reducing them into the form of any one of the above standard integrals I. For the first two (i.e. for (a) and (b)) we discuss on illustrating the suitable examples below. And to integrate the integrals of the type (c), at first we need to express it as,

$$\int \frac{sx + e}{ax^2 + bx + c} dx = p \int \frac{2ax + b}{ax^2 + bx + c} dx + q \int \frac{1}{ax^2 + bx + c} dx \quad (1)$$

where numerator of the first term is the differential coefficient of the denominator. The constants 's' and 'e' are taken as;

$$2ap = s \quad (2)$$

$$\text{and } pb + q = e \quad (3)$$

which are to be solved for the values of p and q . On putting the values of p and q in (1) the integral can easily be integrable. Similar procedure is applied to integrate (d).

Example: 1

Integrate:

$$(a) \int \frac{dx}{x^2 + 6x + 8}$$

$$(b) \int \frac{dx}{\sqrt{x^2 + 6x + 10}}$$

Solution:

$$(a) \text{ Here, } I = \int \frac{dx}{x^2 + 6x + 8} = \int \frac{1}{(x+3)^2 - (1)^2} dx \text{ [which is in the standard form I (d)]}$$

$$\text{or, } I = \frac{1}{2 \times 1} \log \left(\frac{x+3-1}{x+3+1} \right) + c = \frac{1}{2} \log \left(\frac{x+2}{x+4} \right) + c$$

$$\therefore I = \frac{1}{2} \log \left(\frac{x+2}{x+4} \right) + c$$

$$(b) \text{ Here, } I = \int \frac{dx}{\sqrt{x^2 + 6x + 10}} = \int \frac{dx}{\sqrt{x^2 + 3^2 + (1)^2}} \text{ [which is in the standard form I (b)]}$$

$$\text{or, } I = \log [x + 3 + \sqrt{(x+3)^2 + 1^2}] + c$$

$$\therefore I = \log [x + 3 + \sqrt{x^2 + 6x + 10}] + c$$

Example: 2

$$\text{Evaluate: } \int \frac{3x+5}{x^2+4x+20} dx$$

Solution:

$$\text{Let } \int \frac{3x+5}{x^2+4x+20} dx = p \int \frac{2x+4}{x^2+4x+20} dx + q \int \frac{dx}{x^2+4x+20}$$

$$\text{where } 2p = 3 \Rightarrow p = \frac{3}{2}$$

$$\text{and } 4p + q = 5 \Rightarrow 4 \times \frac{3}{2} + q = 5$$

$$\text{or, } q = 5 - 6 = -1$$

$$\therefore p = \frac{3}{2} \text{ and } q = -1$$

For the values of p and q the relation (1) takes the form as,

$$\begin{aligned} \int \frac{3x+4}{(x^2+4x+20)} dx &= \frac{3}{2} \int \frac{2x+4}{x^2+4x+20} dx - \int \frac{dx}{x^2+4x+20} \\ &= \frac{3}{2} \log(x^2+4x+20) - \int \frac{dx}{(x+2)^2+(4)^2} \\ &= \frac{3}{2} \log(x^2+4x+20) - \frac{1}{4} \tan^{-1}\left(\frac{x+2}{4}\right) + c \\ \therefore I &= \int \frac{3x+5}{(x^2+4x+20)} dx \\ &= \frac{3}{2} \log(x^2+4x+20) - \frac{1}{4} \tan^{-1}\left(\frac{x+2}{4}\right) + c \end{aligned}$$

Standard Integral II

In this section, we discuss the integrals which can be evaluated by using the technique of integration by parts. For that we have the formula as,

$\int uv dx = u \int v dx - \int \left[\frac{du}{dx} \int v dx \right] dx$ where u and v are two separate functions. On applying this formula the following standard integrals can be evaluated.

$$(a) \quad \int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx]$$

$$(b) \quad \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2} [a \cos bx + b \sin bx]$$

$$(c) \quad \int \sqrt{a^2+x^2} dx = \frac{x}{2} \sqrt{a^2+x^2} + \frac{a^2}{2} \log(x + \sqrt{a^2+x^2})$$

$$(d) \quad \int \sqrt{a^2-x^2} dx = \frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)$$

$$(e) \quad \int \sqrt{x^2-a^2} dx = \frac{x}{2} \sqrt{x^2-a^2} - \frac{a^2}{2} \log(x + \sqrt{x^2-a^2})$$

One of the above is done for an example.

$$(e) \quad \text{We have, } \int \sqrt{x^2-a^2} dx = \frac{x}{2} \sqrt{x^2-a^2} - \frac{a^2}{2} \log(x + \sqrt{x^2-a^2})$$

Proof:

$$\text{L.H.S.} = \int \sqrt{x^2-a^2} dx$$

Integrating by parts, we have,

$$\int \sqrt{x^2-a^2} dx = \sqrt{x^2-a^2} \int dx - \int \left[\frac{d\sqrt{x^2-a^2}}{dx} \int dx \right] dx$$

$$\text{or, } \int \sqrt{x^2-a^2} dx = \sqrt{x^2-a^2} \cdot x - \int \frac{2x}{2\sqrt{x^2-a^2}} \cdot x dx$$

$$\begin{aligned}
 &= x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}} dx \\
 &= x\sqrt{x^2-a^2} - \int \frac{(x^2-a^2)+a^2}{\sqrt{x^2-a^2}} dx
 \end{aligned}$$

$$\text{or, } \int \sqrt{x^2-a^2} dx = x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2} dx - a^2 \int \frac{1}{\sqrt{x^2-a^2}} dx$$

$$\text{or, } 2. \int \sqrt{x^2-a^2} dx = x\sqrt{x^2-a^2} - a^2 \log [x + \sqrt{x^2-a^2}]$$

$$\therefore \int \sqrt{x^2-a^2} dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2} \log [x + \sqrt{x^2-a^2}]$$

Note:

The integrals, (a) $\int \sqrt{ax^2+bx+c} dx$, (b) $\int (sx+e)\sqrt{ax^2+bx+c} dx$ can be evaluated on reducing them to any one of the standard integrals II, which are discussed with the suitable examples as below.

Example: 3

Evaluate:

$$(a) \int \sqrt{5-2x+x^2} dx$$

$$(b) \int (2x+1)\sqrt{4x^2+20x+21} dx$$

Solution:

$$\begin{aligned}
 (a) \quad \text{Here, } I &= \int \sqrt{5-2x+x^2} dx \\
 &= \int \sqrt{4+(x-1)^2} dx = \int \sqrt{(2)^2+(x-1)^2} dx
 \end{aligned}$$

which is in the standard form of II (c)

$$\therefore I = \frac{(x-1)}{2} \sqrt{2^2+(x-1)^2} + \frac{2^2}{2} \log [x-1 + \sqrt{2^2+(x-1)^2}] + c$$

$$\therefore I = \frac{1}{2}(x-1)\sqrt{5-2x+x^2} + 2 \log [x-1 + \sqrt{5-x+x^2}] + c$$

$$(b) \quad \text{Here, } I = \int (2x+1)\sqrt{4x^2+20x+21} dx$$

It can be expressed as;

$$\begin{aligned}
 &\int (2x+1)\sqrt{4x^2+20x+21} dx \\
 &= p \int (8x+20)\sqrt{4x^2+20x+21} dx + q \int \sqrt{4x^2+20x+21} dx \quad (1)
 \end{aligned}$$

$$\text{where } 8p = 2 \Rightarrow p = \frac{1}{4}$$

$$\text{and } 20p + q = 1$$

$$\text{or, } 20 \times \frac{1}{4} + q = 1$$

$$\text{or, } q = 1 - 5 = -4$$

$$\therefore p = \frac{1}{4} \text{ and } q = -4$$

For the values of p and q the relation (1) takes the form as,

$$\begin{aligned} I &= \frac{1}{4} \int (8x + 20) \sqrt{4x^2 + 20x + 21} \, dx - 4 \int \sqrt{4x^2 + 20x + 21} \, dx \\ &= \frac{1}{4} \times \frac{2}{3} (4x^2 + 20x + 21)^{3/2} + c_1 - 4 \int \sqrt{(2x + 5)^2 - 2^2} \, dx \\ &= \frac{1}{6} (4x^2 + 20x + 21)^{3/2} + c_1 - \frac{4}{2} \left[\frac{2x + 5}{2} \sqrt{(2x + 5)^2 - 2^2} \right] - \frac{4}{2} \log [2x + 5 + \sqrt{(2x + 5)^2 - 2^2}] + c_2 \\ &= \frac{1}{6} (4x^2 + 20x + 21)^{3/2} - 2 \left[\frac{2x + 5}{2} \sqrt{4x^2 + 20x + 21} - 2 \log (2x + 5) + \sqrt{4x^2 + 20x + 21} \right] + c_1 + c_2 \\ \therefore I &= \frac{1}{6} (4x^2 + 20x + 21)^{3/2} - (2x + 5) \sqrt{4x^2 + 20x + 21} + 4 \log [2x + 5 + \sqrt{4x^2 + 20x + 21}] + c, \end{aligned}$$

where $c = c_1 + c_2$

Standard Integrals Involving Trigonometric Functions

The various fundamental techniques on integrating some of the special trigonometric functions are as below:

$$(a) \quad \int \operatorname{cosec} x \, dx = \int \frac{1}{\sin x} \, dx = \int \frac{dx}{2 \sin \frac{x}{2} \cos \frac{x}{2}}$$

$$= \frac{1}{2} \int \frac{1}{\sin \frac{x}{2} \cos \frac{x}{2}} \times \frac{\sec^2 \frac{x}{2}}{\sec^2 \frac{x}{2}} \, dx = \frac{1}{2} \int \frac{\sec^2 \frac{x}{2}}{\tan \frac{x}{2}} \, dx = \log \left(\tan \frac{x}{2} \right) + c$$

$$\therefore \int \operatorname{cosec} x \, dx = \log \left(\tan \frac{x}{2} \right) + c$$

$$(b) \quad \int \sec x \, dx = \int \frac{1}{\cos x} \, dx = \int \frac{1}{\sin \left(\frac{\pi}{2} + x \right)} \, dx$$

$$= \frac{1}{2} \int \frac{dx}{\sin \left(\frac{\pi}{4} + \frac{x}{2} \right) \cos \left(\frac{\pi}{4} + \frac{x}{2} \right)} = \frac{1}{2} \int \frac{\sec^2 \left(\frac{\pi}{4} + \frac{x}{2} \right)}{\tan \left(\frac{\pi}{4} + \frac{x}{2} \right)} \, dx \quad \left[\because \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} \right]$$

$$= \log \left[\tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right] + c \quad [\because \text{Multiply by } \cos \left(\frac{\pi}{4} + \frac{x}{2} \right) \text{ on numerator and denominator both}]$$

$$\therefore \int \sec x \, dx = \log \left[\tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right] + c$$

Standard Integrals III

In this section, we discuss about the standard integrals evolving the trigonometric functions. The various fundamental techniques to integrate some of the standard trigonometrical functions are as below:

(a) $\int \frac{dx}{a + b \cos x}$

(b) $\int \frac{dx}{a + b \sin x}$

(c) $\int \frac{dx}{a \sin x + b \cos x}$

To integrate the first two (i.e. a and b) we will discuss later with examples. Now to integrate the third one we provide $a = r \cos \beta$ and $b = r \sin \beta$ so that $a^2 + b^2 = r^2 (\sin^2 \beta + \cos^2 \beta) = r^2 \Rightarrow r = \sqrt{a^2 + b^2}$ and $\tan \beta = \frac{b}{a} \Rightarrow \beta = \tan^{-1} \left(\frac{b}{a} \right)$.

$$\text{Now, we have } I = \int \frac{dx}{a \sin x + b \cos x} = \frac{1}{r} \int \frac{dx}{\sin x \cos \beta + \cos x \sin \beta}$$

$$\text{or, } I = \frac{1}{r} \int \frac{dx}{\sin(x + \beta)} = \frac{1}{r} \int \operatorname{cosec}(x + \beta) dx$$

$$\text{or, } I = \frac{1}{r} \log \left[\tan \left(\frac{x + \beta}{2} \right) \right] + c$$

$$\therefore I = \frac{1}{\sqrt{a^2 + b^2}} \log \left[\tan \left(\frac{x + \beta}{2} \right) \right] + c$$

Note:

It can also be evaluated on reducing it to the form of any one of the standard integrals I on putting

$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$ and $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$ then we take $\tan \frac{x}{2} = s$. Now, it will easily be converted in the form of the standard integrals I .

Example: 4

Integrate: (a) $\int \frac{dx}{1 - 3 \sin x}$

(b) $\int \frac{dx}{3 \sin x + 4 \cos x}$

Solution:

$$(a) \text{ Here, } I = \int \frac{dx}{1 - 3 \sin x} = \int \frac{dx}{\left(\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} \right) - 6 \sin \frac{x}{2} \cos \frac{x}{2}}$$

$$\begin{aligned} \text{or, } I &= \int \frac{dx}{\left(\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} \right) - 6 \sin \frac{x}{2} \cos \frac{x}{2}} \times \frac{\sec^2 \frac{x}{2}}{\sec^2 \frac{x}{2}} \\ &= \int \frac{dx}{\left(\tan^2 \frac{x}{2} + 1 \right) - 6 \tan \frac{x}{2} \cdot 1} \times \sec^2 \frac{x}{2} \\ &= \int \frac{\sec^2 \frac{x}{2}}{\tan^2 \frac{x}{2} - 6 \tan \frac{x}{2} + 1} dx \end{aligned}$$

Putting $\tan \frac{x}{2} = y$. Differentiating we get

$$\frac{1}{2} \sec^2 \frac{x}{2} dx = dy \Rightarrow \sec^2 \frac{x}{2} dx = 2dy$$

$$\begin{aligned} \therefore I &= \int \frac{2dy}{y^2 - 6y + 1} = 2 \int \frac{dy}{(y-3)^2 - (2\sqrt{2})^2} \\ &= \frac{2 \times 1}{2 \times 2\sqrt{2}} \log \left(\frac{y-3-2\sqrt{2}}{y-3+2\sqrt{2}} \right) + c \\ \therefore I &= \frac{1}{2\sqrt{2}} \log \left[\frac{\tan \frac{x}{2} - (3+2\sqrt{2})}{\tan \frac{x}{2} - (3-2\sqrt{2})} \right] \end{aligned}$$

(b) Here, $I = \int \frac{dx}{3 \sin x + 4 \cos x}$

Let $3 = r \cos \theta$ and $4 = r \sin \theta$

so that $r^2 = 16 + 9 = 25 \Rightarrow r = 5$

and $\tan \theta = \frac{4}{3}$

or, $\theta = \tan^{-1} \left(\frac{4}{3} \right)$

$$\therefore I = \int \frac{dx}{r \cos \theta \sin x + r \sin \theta \cos x} = \frac{1}{r} \int \frac{dx}{\sin(x + \theta)}$$

or, $I = \frac{1}{r} \int \operatorname{cosec}(x + \theta) dx = \frac{1}{r} \log \tan \left(\frac{x + \theta}{2} \right) + c \quad \left[\because \int \operatorname{cosec}(x + \theta) dx = \log \tan \frac{x}{2} + c \right]$

$$\therefore I = \frac{1}{5} \log \tan \frac{1}{2} \left[x + \tan^{-1} \left(\frac{4}{3} \right) \right] + c$$

$$\therefore I = \frac{1}{5} \log \tan \left[\frac{x}{2} + \frac{1}{2} \tan^{-1} \left(\frac{4}{3} \right) \right] + c$$

EXERCISE - 3 (II)

Difficulty Level: A

1. Evaluate:

(a) $\int \frac{dx}{x^2 + 49}$

(b) $\int \frac{dx}{4x^2 - 9}$

(c) $\int \frac{dx}{1 + x - x^2}$

(d) $\int \frac{dx}{\sqrt{x^2 - 16}}$

2. Evaluate:

(a) $\int \frac{dx}{2 + \cos x}$

(b) $\int \frac{dx}{3 - \cos x}$

(c) $\int \frac{dx}{2 + \sin x}$

Difficulty Level: B**3. Integrate:**

(a) $\int \frac{dx}{4x^2 + 24x + 45}$

(b) $\int \frac{(4x+3)}{4x^2 + 12x + 5} dx$

(c) $\int \frac{x dx}{x^2 + 2x + 2}$

(d) $\int \frac{dx}{\sqrt{2ax + x^2}}$

(e) $\int \frac{(3x+2) dx}{\sqrt{2x^2 + 5x + 4}}$

(f) $\int \sqrt{25 - 9x^2} dx$

(g) $\int \sqrt{3 - 2x - x^2} dx$

(h) $\int \sqrt{9x^2 + 24x + 25} dx$

(i) $\int (2x+5) \sqrt{4x^2 + 12x + 45} dx$

4. Evaluate:

(a) $\int \frac{dx}{\sin x + \cos x}$

(b) $\int \frac{dx}{2\sin x + 3\cos x}$

ANSWERS

1. (a) $\frac{1}{7} \tan^{-1} \left(\frac{x}{7} \right) + c$

(b) $\frac{1}{12} \log \left(\frac{2x-3}{2x+3} \right) + c$

(c) $\frac{1}{\sqrt{5}} \log \left(\frac{\sqrt{5}-1+2x}{\sqrt{5}+1-2x} \right) + c$

(d) $\log [x + \sqrt{x^2 - 16}] + c$

2. (a) $\frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{\tan \frac{x}{2}}{\sqrt{3}} \right) + c$

(b) $\frac{1}{\sqrt{2}} \tan^{-1} \left(\sqrt{2} \tan \frac{x}{2} \right) + c$

(c) $\frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2 \tan \frac{x}{2} + 1}{\sqrt{3}} \right) + c$

3. (a) $\frac{1}{6} \tan^{-1} \left(\frac{2x+6}{3} \right) + c$

(b) $\frac{1}{2} \log (4x^2 + 12x + 5) - \frac{3}{8} \log \frac{2x+1}{2x+5} + c$

(c) $\frac{1}{2} \log (x^2 + 2x + 2) - \tan^{-1} (x+1) + c$

(d) $\log [(x+4) + \sqrt{2ax + x^2}] + c$

(e) $\frac{3}{2} \sqrt{2x^2 + 5x + 4} - \frac{7}{4\sqrt{2}} \log \left[x + \frac{5}{4} + \sqrt{x^2 + \frac{5x}{2} + 2} \right] + c$

(f) $\frac{1}{6} \left[3x \sqrt{25 - 9x^2} + 25 \sin^{-1} \left(\frac{3x}{5} \right) \right] + c$

(g) $\frac{1}{2} (x+1) \sqrt{3 - 2x - x^2} + 2 \sin^{-1} \left(\frac{x+1}{2} \right) + c$

(h) $\frac{1}{6} (3x+4) \sqrt{9x^2 + 24x + 25} + \frac{3}{2} \log [3x+4 + \sqrt{9x^2 + 24x + 25}] + c$

(i) $\frac{1}{6} (4x^2 + 12x + 45)^{3/2} + \frac{1}{2} (2x+3) \sqrt{4x^2 + 12x + 45} + 18 \log [2x+3 + \sqrt{4x^2 + 12x + 45}] + c$

4 (a) $\frac{1}{\sqrt{2}} \log \tan \left(\frac{x}{2} + \frac{\pi}{8} \right) + c$

(b) $\frac{1}{\sqrt{13}} \log \tan \left[\frac{x}{2} + \frac{1}{2} \tan^{-1} \left(\frac{3}{2} \right) \right] + c$

3.4 Basic Idea of Curve Sketching

Graph is one of the various ways of representing a function. In this chapter, we sketch the graph of a function with the help of the different characteristics which the graph satisfies.

Characteristics of Curves

1. **Origin:** If $(0, 0)$ satisfies the equation, then the graph of the equation passes through the origin. The equation $y = x^2 - 2x$ is satisfied by $(0, 0)$. So, the graph passes through the origin. But $y = x^2 - 2x + 3$ is not satisfied by $(0, 0)$. So, the graph does not pass through the origin.
2. **Intercepts:** For x -intercepts, we put $y = 0$ in the given equation to get the corresponding x -values.

Putting $y = 0$ in

$$x^2 + y^2 = 4$$

(1)

gives $x^2 = 4$

or, $x = \pm 2$

This means that the graph of (1) cuts x -axis at the points $(\pm 2, 0)$.

For y -intercepts, we put $x = 0$ in the equation to get the corresponding y -values.

Putting $x = 0$ in equation (1) gives

$$y^2 = 4$$

or, $y = \pm 2$

This means that the graph of (1) cuts y -axis at the points $(0, \pm 2)$.

3. **Even Function:** The function $f: A \rightarrow B$ is called even function if $f(-x) = f(x)$ for all $x \in A$.

Examples:

(a) $y = f(x) = \cos x$

Since $f(-x) = \cos(-x) = \cos x = f(x)$, so $y = f(x)$ is even function.

(b) $y = f(x) = x^4 + 2x^2$

Since $f(-x) = (-x)^4 + 2(-x)^2 = x^4 + 2x^2 = f(x)$, so it is an even function.

(c) $y = f(x) = x^3 - 2x$

Here,

$$\begin{aligned} f(-x) &= (-x)^3 - 2(-x) \\ &= -x^3 + 2x \\ &\neq f(x) \end{aligned}$$

So, this is not an even function.

Thus all algebraic functions with even powers are even functions.

4. **Odd Function:** The function $f: A \rightarrow B$ is called odd function if $f(-x) = -f(x)$ for all $x \in A$.

Examples:

(a) $y = f(x) = \sin x$

Since $f(-x) = \sin(-x) = -\sin x = -f(x)$, so it is an odd function.

(b) $y = f(x) = x^3$

Since $f(-x) = (-x)^3 = -x^3 = -f(x)$, so f is an odd function.

(c) $y = f(x) = x^5 - \sin x$

Since $f(-x) = (-x)^5 - \sin(-x)$

$$\begin{aligned}
&= -x^5 + \sin x \\
&= -(x^5 - \sin x) \\
&= -f(x), \text{ so } f \text{ is an odd function.}
\end{aligned}$$

Thus all algebraic functions with odd powers are odd functions.

5. Symmetry:

- (a) The graph of the function $y = f(x)$ is symmetric about the x -axis if replacing y by $-y$ gives the same equation. Consider the equation

$$y^2 = 4x$$

Replacing y by $-y$ in this equation gives

$$(-y)^2 = 4x$$

$$\text{or, } y^2 = 4x$$

which is same as the original equation. So the graph of $y^2 = 4x$ is symmetrical about the x -axis.

- (b) The graph of an equation is symmetric about the y -axis if replacing x by $-x$ gives the same equation. Consider the equation.

$$y = x^2$$

Replacing x by $-x$ in this equation gives

$$y = (-x)^2$$

or, $y = x^2$ which is same as the original equation. So, the graph of $y = x^2$ is symmetrical about the y -axis.

- (c) The graph of an equation is symmetric about the origin if replacing x by $-x$ and y by $-y$ produces the same equation. Consider the equation

$$x^2 + y^2 = 4$$

Replacing x by $-x$ and y by $-y$ gives

$$(-x)^2 + (-y)^2 = 4$$

$$\text{or, } x^2 + y^2 = 4$$

This means that the graph of the equation $x^2 + y^2 = 4$ is symmetrical about the origin.

Monotonicity of a Function

A function $y = f(x)$ is said to be an increasing in an open interval (a, b) if

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \text{ for all } x_1, x_2 \in (a, b).$$

For example: Let $y = f(x) = x^2$ on $(0, \infty)$. Then

$$1 < 2 \Rightarrow f(1) < f(2) \Rightarrow 1 < 4$$

$$3 < 4 \Rightarrow f(3) < f(4) \Rightarrow 9 < 16 \text{ and so on.}$$

Thus, the graph of $y = f(x) = x^2$ is increasing on $(0, \infty)$.

A function $y = f(x)$ is said to be decreasing in the open interval (a, b) if

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \text{ for all } x_1, x_2 \in (a, b).$$

For example: Let $y = f(x) = x^2$ on $(-\infty, 0)$. Then

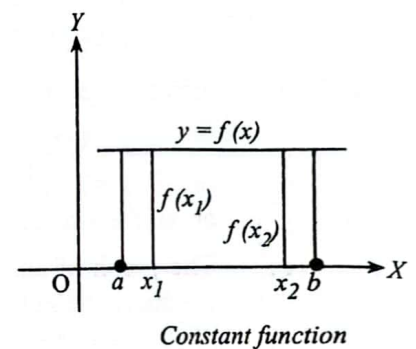
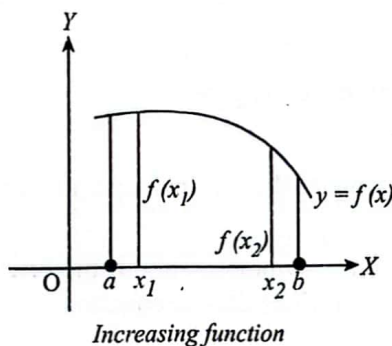
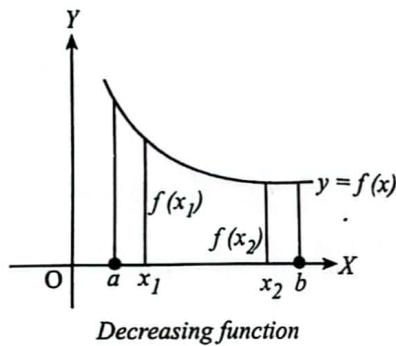
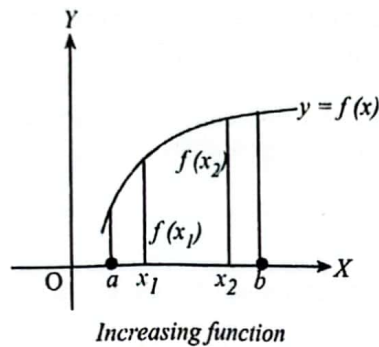
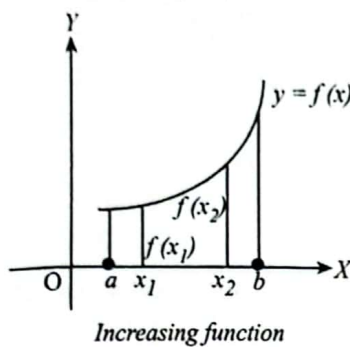
$$-2 < -1 \Rightarrow (-2)^2 > (-1)^2 \Rightarrow 4 > 1$$

$$\therefore f(-2) > f(-1)$$

Also, $-4 > -10 \Rightarrow f(-4) < f(-10)$ and so on. Thus, the graph of $y = f(x) = x^2$ is a decreasing function on $(-\infty, 0)$.

A function $y = f(x)$ is called monotonic if it is either increasing or decreasing on the given interval. In other words, increasing and decreasing functions are also called monotonic functions.

Note that $y = f(x)$ is constant function in interval (a, b) if $x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$



7. Periodicity: A function f which satisfies the condition

$$f(x + k) = f(x)$$

for all $x \in A$, where $f: A \rightarrow B$, and $k > 0$ is called periodic function. The smallest value of k is known as the fundamental period of the function.

Examples:

(a) The functions

$$f(x) = \sin x$$

and $f(x) = \cos x$

are periodic functions with period 2π because

$$f(x + 2\pi) = \sin(x + 2\pi) = \sin x$$

and $f(x + 2\pi) = \cos(x + 2\pi) = \cos x$

Also, we can write

$$f(x + 2n\pi) = \sin(x + 2n\pi) = \sin x$$

$$f(x + 2n\pi) = \cos(x + 2n\pi) = \cos x \text{ for } n = \pm 1, \pm 2, \pm 3, \dots$$

(b) Since $\tan(x + \pi) = \tan x$

and $\cot(x + \pi) = \cot x$,

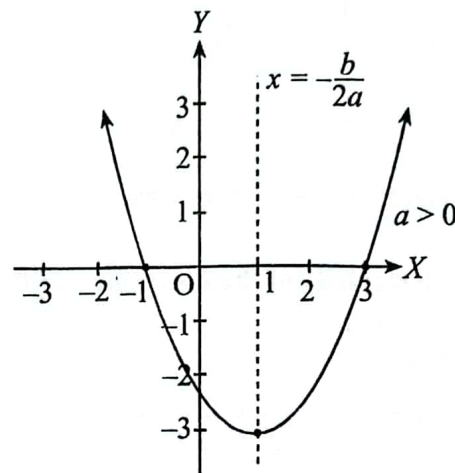
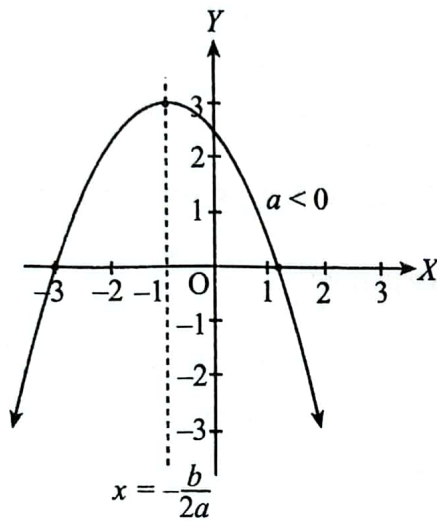
So, $f(x) = \tan x$ and $f(x) = \cot x$ are also periodic functions with fundamental period π .

8. Quadratic function

Let the quadratic function be

$$y = f(x) = ax^2 + bx + c \quad (a \neq 0)$$

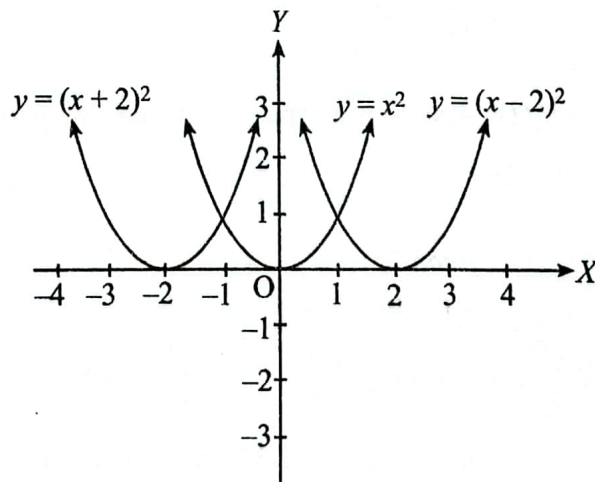
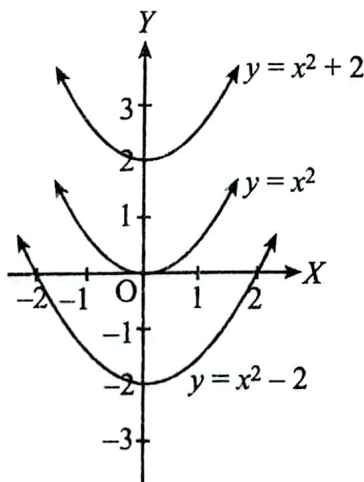
- (a) The graph of this function is a parabola.
- (b) If $a > 0$, the parabola opens upward.
- (c) If $a < 0$, the parabola opens downward.
- (d) The coordinates of the vertex are $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.
- (e) The line of symmetry is $x = -\frac{b}{2a}$.
- (f) The maximum or minimum value is $f\left(-\frac{b}{2a}\right)$.



9. Transformation of Graphs

Let $y = f(x)$ and $c > 0$. Then

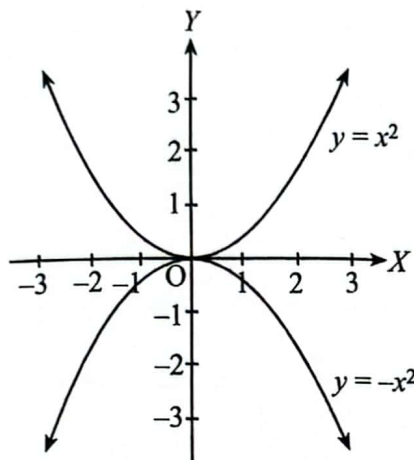
- (a) to draw the graph of $y = f(x) + c$ shift the graph of $y = f(x)$ c units up.
- (b) to draw the graph of $y = f(x) - c$ shift the graph of $y = f(x)$ c units down.
- (c) to draw the graph of $y = f(x - c)$ shift the graph of $y = f(x)$ c units to the right.
- (d) to draw the graph of $y = f(x + c)$ shift the graph of $y = f(x)$ c units to the left.



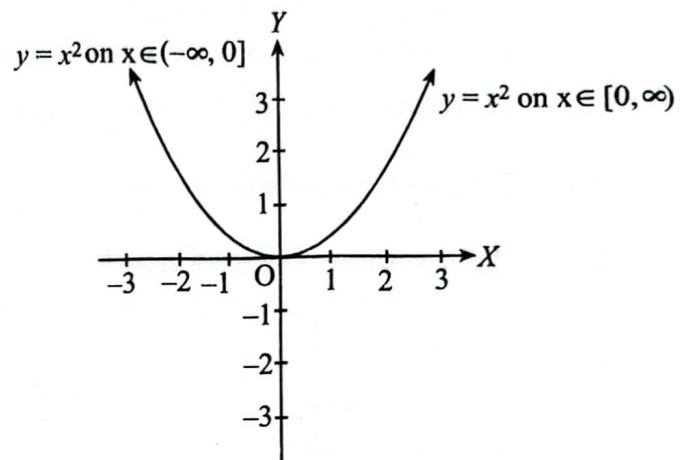
10. Reflection

The graph of $y = f(-x)$ is the reflection of the graph of $y = f(x)$ about y -axis.

The graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$ about x -axis.



Reflection of $y = x^2$ about x -axis



Reflection of $y = x^2$ on; $[0, \infty)$ about y -axis

Examples:**1. Test the periodicity of the functions and find the period:**

(a) $f(x) = \cos 2x$

Now,

$$\begin{aligned} f(x) &= \cos 2x = \cos (2x + 2\pi) \\ &= \cos 2(x + \pi) \\ &= f(x + \pi) \end{aligned}$$

$$\therefore f(x + \pi) = f(x)$$

This shows that $f(x) = \cos 2x$ is a periodic function with period π .

(b) $f(x) = \sin bx$

Now, $f(x) = \sin bx$

$$\begin{aligned} &= \sin (2\pi + bx) \\ &= \sin b \left(x + \frac{2\pi}{b} \right) \\ &= f \left(x + \frac{2\pi}{b} \right) \end{aligned}$$

This shows that $f(x) = \sin bx$ is a periodic function with period $\frac{2\pi}{b}$.

(c) $f(x) = \tan 2x$

Now,

$$\begin{aligned} f(x) &= \tan 2x \\ &= \tan (\pi + 2x) \\ &= \tan 2 \left(x + \frac{\pi}{2} \right) = f \left(x + \frac{\pi}{2} \right) \end{aligned}$$

Thus, $f(x) = \tan 2x$ is a periodic function with period $\frac{\pi}{2}$.

2. Show that $f(x) = 4 - 3x$ is decreasing for all $x \in R$.

Let $x_1, x_2 \in R$. Then

$$\begin{aligned} x_1 > x_2 &\Rightarrow -3x_1 < -3x_2 \\ &\Rightarrow 4 - 3x_1 < 4 - 3x_2 \\ &\Rightarrow f(x_1) < f(x_2) \end{aligned}$$

This shows that $f(x)$ is decreasing for all $x \in R$.

3. Show that $f(x) = 3x + 4$ is increasing for all $x \in R$.

Let $x_1, x_2 \in R$, Then

$$\begin{aligned} x_1 > x_2 &\Rightarrow 3x_1 > 3x_2 \\ &\Rightarrow 3x_1 + 4 > 3x_2 + 4 \\ &\Rightarrow f(x_1) > f(x_2) \end{aligned}$$

So, $f(x)$ is increasing for all $x \in R$.

4. Show that $f(x) = e^{2x+1}$ is increasing for all $x \in R$.

Let $x_1, x_2 \in R$, Then

$$\begin{aligned} x_1 > x_2 &\Rightarrow 2x_1 > 2x_2 \\ &\Rightarrow 2x_1 + 1 > 2x_2 + 1 \\ &\Rightarrow e^{2x_1+1} > e^{2x_2+1} \\ &\Rightarrow f(x_1) > f(x_2) \end{aligned}$$

Thus, $f(x) = e^{2x+1}$ is increasing for all $x \in R$.

5. Examine the symmetricity of the function $f(x) = e^x + e^{-x}$.

Now,

$$\begin{aligned} f(-x) &= e^{-x} + e^{-(-x)} \\ &= e^{-x} + e^x \\ &= f(x) \end{aligned}$$

This shows that $f(x) = e^x + e^{-x}$ is symmetric about y-axis.

6. Examine whether the function.

$$f(x) = \sqrt{1+x^2} - \sqrt{1-x^2}$$

is even, odd or neither.

Now,

$$\begin{aligned} f(-x) &= \sqrt{1+(-x)^2} - \sqrt{1-(-x)^2} \\ &= \sqrt{1+x^2} - \sqrt{1-x^2} \\ &= f(x) \end{aligned}$$

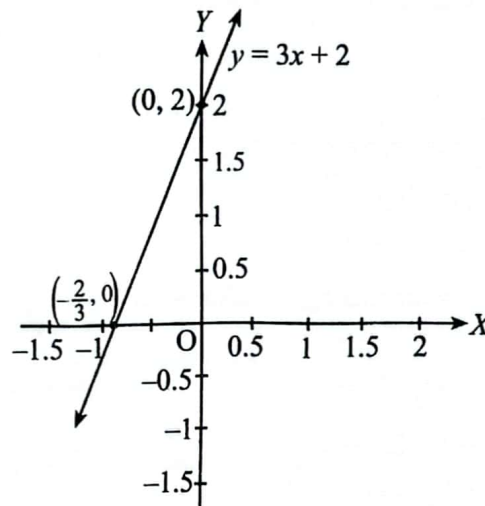
So, $f(x)$ is an even function.

7. Sketch the graph of the function $y = f(x) = 3x + 2$.

(a) When $x = 0$, $y = 2$. So, the graph does not pass through the origin. But it crosses y-axis at $(0, 2)$.

- (b) When $y = 0$, $x = -\frac{2}{3}$. So, the graph cuts x -axis at $(-\frac{2}{3}, 0)$.
- (c) When $x > 0$, $y > 2$ and when $x < -\frac{2}{3}$, $y < 0$.

The graph of $f(x)$ with the above characteristics is as follows:



8. Sketch the graph of $y = f(x) = x^2 - 6x + 5$.

Comparing given function with

$$y = f(x) = ax^2 + bx + c \text{ gives}$$

$$a = 1, b = -6, c = 5$$

- (a) Since $a > 0$, so the parabola opens upward.
- (b) When $x = 0$, $y = 5$, So, the graph does not pass through the origin. It cuts y -axis at $(0, 5)$.
- (c) When $y = 0$, $x^2 - 6x + 5 = 0$
or, $(x - 1)(x - 5) = 0$
or, $x = 1, 5$

Thus, the graph intersects x -axis at $(1, 0)$ and $(5, 0)$.

- (d) x -coordinate of the vertex

$$= -\frac{b}{2a} = -\frac{-6}{2 \times 1} = 3$$

y -coordinate of the vertex

$$= f\left(-\frac{b}{2a}\right) = f(3) = 3^2 - 6(3) + 5 = -4$$

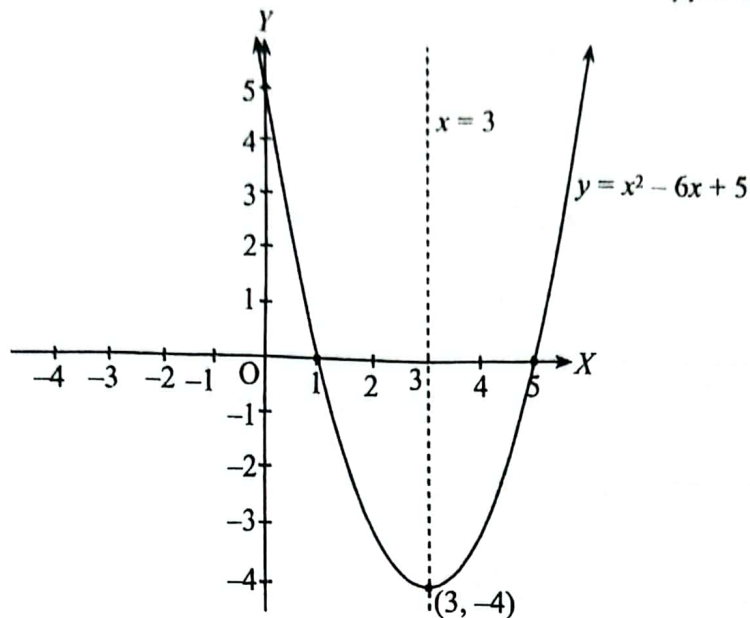
$$\therefore \text{Vertex} = (3, -4)$$

- (e) The graph is symmetric about the line

$$x = x\text{-coordinate of the vertex}$$

$$\text{or, } x = 3$$

The graph of the given function with the above characteristics is given below:

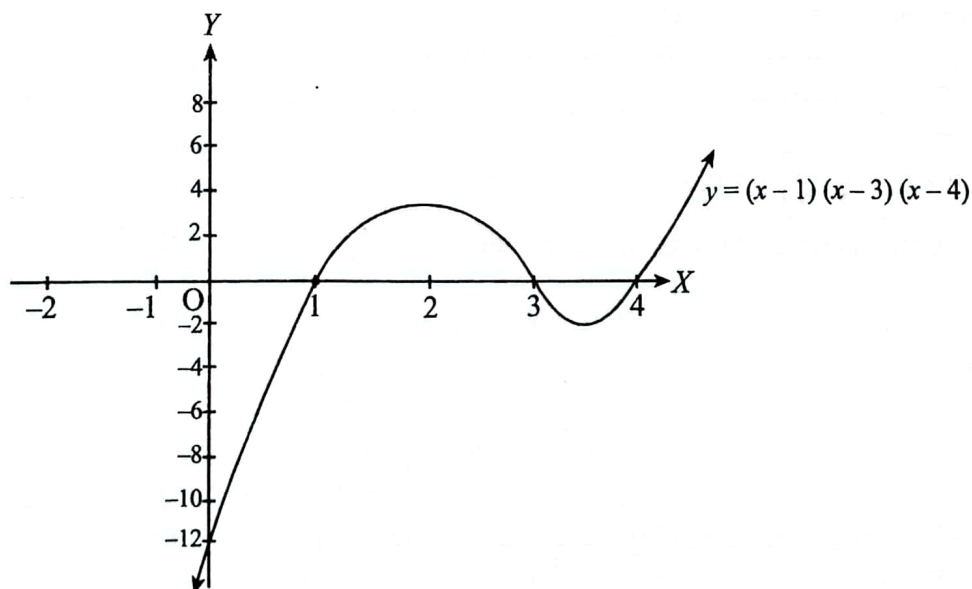


9. Sketch the graph of $y = f(x) = (x - 1)(x - 3)(x - 4)$.

The characteristics of the given function are:

- (a) When $x = 0$, $y = -12$ So, the graph intersects y -axis at $(0, -12)$. The graph does not pass through the origin.
- (b) When $y = 0$, $x = 1, 3, 4$ So, the graph intersects x -axis at $(1, 0)$, $(3, 0)$ and $(4, 0)$.
- (c) When $x < 1$, $y < 0$. So, when x decreases y also decreases.
- (d) When $x > 4$, $y > 0$ So, when x increases, y also increases.
- (e) When $1 < x < 3$, $y > 0$ and when $3 < x < 4$, $y < 0$. This means that the graph lies above x -axis when $1 < x < 3$ and below x -axis when $3 < x < 4$.

The graph of the given function with the above properties is given below:

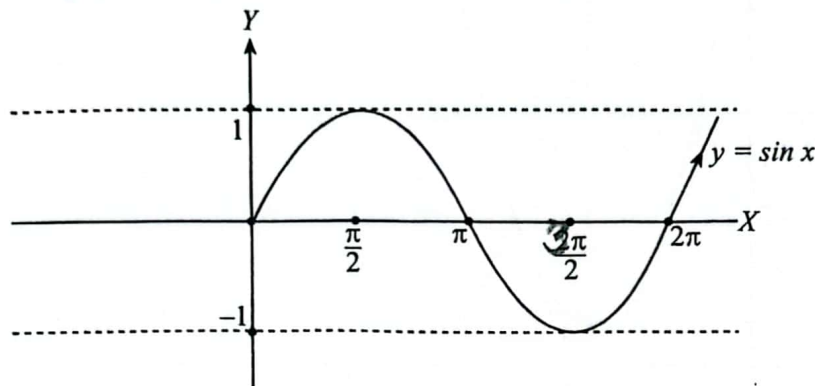


10. Sketch the graph of $y = f(x) = \sin x$ on $[0, 2\pi]$.

- (a) When $x = 0$, $y = 0$ So, the graph passes through the origin.
- (b) $f(-x) = \sin(-x) = -\sin x = -f(x)$, So, the graph is symmetric about the origin.
- (c) When $y = 0$, $x = 0, \pi, 2\pi$.

- (d) The graph rises to 1 on $\left[0, \frac{\pi}{2}\right]$, falls down to -1 on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and again rises to 0 on $\left[\frac{3\pi}{2}, 2\pi\right]$.

The graph of the given function with the above properties is as follows:



11. Sketch the graph of $y = \cos x$ on $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

- (a) When $x = 0, y = 1$ So, the graph cuts y -axis at $(0, 1)$.

- (b) When $y = 0$, we get,
 $\cos x = 0$

$$= \cos\left(\pm \frac{\pi}{2}\right)$$

$$\text{or, } x = \pm \frac{\pi}{2}$$

So, the graph intersects x -axis at $\left(-\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 0\right)$.

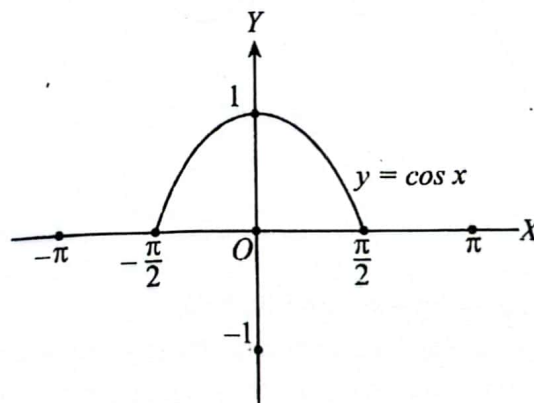
- (c) $f(-x) = \cos(-x) = \cos x = f(x)$

So, $f(x)$ is even function and is symmetric about the y -axis.

- (d) For $-\frac{\pi}{2} \leq x \leq 0$ the graph is increasing and for $0 \leq x \leq \frac{\pi}{2}$, the graph is decreasing.

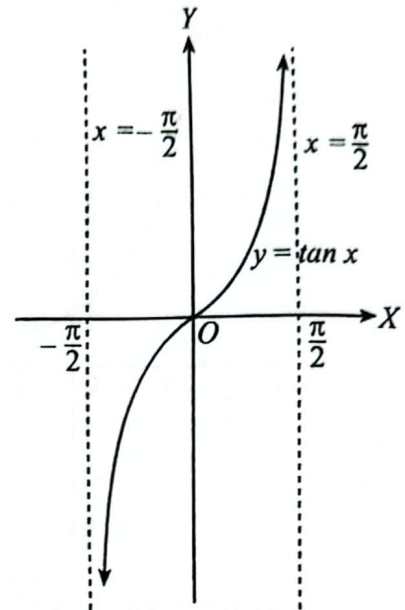
- (e) For all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \geq 0$. So, the graph lies above x -axis.

With the above mentioned characteristics we have the following graph:



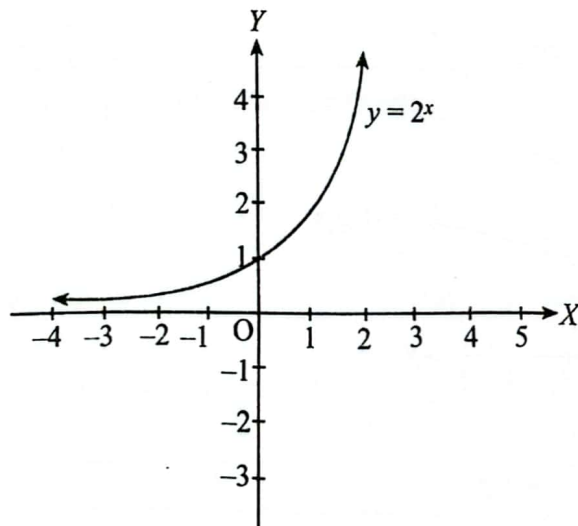
12. Sketch the graph of $y = f(x) = \tan x$ on $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

- (a) When $x = 0$, $y = 0$ So, the graph passes through the origin.
- (b) $f(-x) = \tan(-x) = -\tan x = -f(x)$
So, $f(x)$ is an odd function and the graph is symmetric about the origin.
- (c) For $-\frac{\pi}{2} < x < 0$, $y < 0$, So, the graph is below x -axis. For $0 < x < \frac{\pi}{2}$, $y > 0$, So, the graph is above x -axis.
- (d) For all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the graph is increasing.



13. Sketch the graph of $y = f(x) = 2^x$.

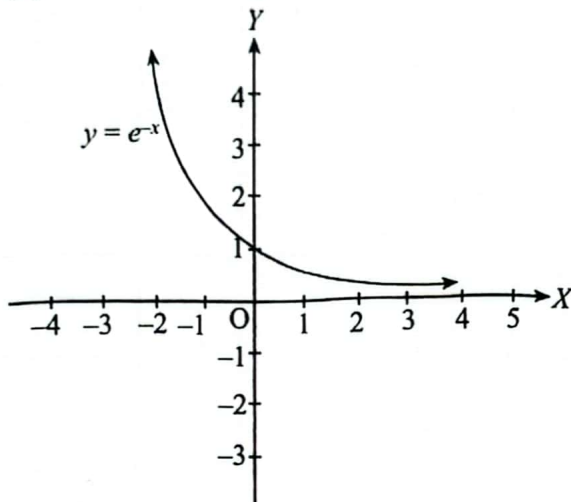
- (a) When $x = 0$, $y = 1$ So, the graph does not pass through the origin. But it intersects y -axis at $(0, 1)$.
- (b) For all $x_1, x_2 \in \mathbb{R}$,
 $x_1 > x_2 \Rightarrow 2^{x_1} > 2^{x_2}$
 $\Rightarrow f(x_1) > f(x_2)$
So, the graph is increasing.
- (c) There is no symmetry.
- (d) When $x \rightarrow -\infty$, $y \rightarrow 0$.



14. Sketch the graph of $y = f(x) = e^{-x}$.

- (a) When $x = 0$, $y = 1$ So, the graph does not pass through the origin. But it cuts y -axis at $(0, 1)$.
- (b) For all $x_1, x_2 \in \mathbb{R}$,
 $x_1 > x_2 \Rightarrow -x_1 < -x_2$
 $\Rightarrow e^{-x_1} < e^{-x_2}$
 $\Rightarrow f(x_1) < f(x_2)$
So, the graph is decreasing.

- (c) There is no symmetry.
 (d) When $x \rightarrow \infty, y \rightarrow 0$.



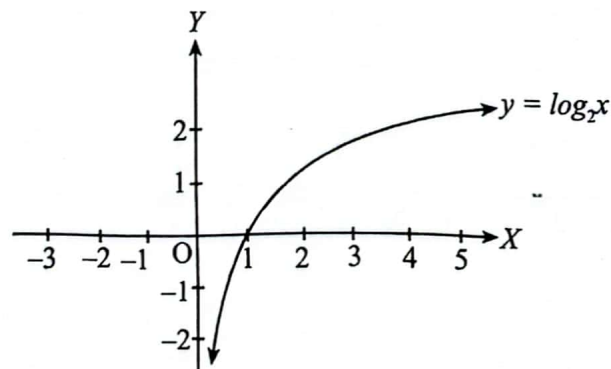
15. Sketch the graph of $y = f(x) = \log_2 x$.

We have

$$y = \log_2 x$$

or, $x = 2^y$

- (a) When $y = 0, x = 1$ So, the graph cuts x-axis at $(1, 0)$.
 (b) When $x \rightarrow 0, y \rightarrow -\infty$.
 (c) For all $x_1, x_2 \in (0, \infty)$,
 $x_1 > x_2 \Rightarrow \log_2 x_1 > \log_2 x_2$
 $\Rightarrow f(x_1) > f(x_2)$
 So, the graph is always increasing.
 (d) There is no symmetry.



EXERCISE 3 (III)**Difficulty Level: A**

1. Examine whether the following functions are even, odd or neither:

(a) $f(x) = x^3 + \sin x$

(b) $f(x) = 5x^2 + \sin x$

(c) $f(x) = x^2 + \cos x + 3$

(d) $f(x) = |x - 3|$

(e) $f(x) = 10^x - 10^{-x}$

2. Test for symmetry of the following functions:

(a) $f(x) = \sqrt{4 - x^2}$

(b) $f(x) = e^x + e^{-x}$

(c) $f(x) = x^3 - x$

(d) $f(x) = \frac{1}{x^3} + \frac{1}{x}$

(e) $f(x) = \sin x + \cos x$

3. Test for periodicity of the following functions and find their periods:

(a) $f(x) = \sin 2x$

(c) $f(x) = \sin x + \cos x$

(c) $f(x) = \sin x + \tan x$

(d) $f(x) = \cos \pi x$

(e) $f(x) = \tan \frac{1}{4}x$

4. Show that $f(x) = 2x + 7$ is increasing for all $x \in \mathbb{R}$.

5. Show that $f(x) = 7 - 2x$ is decreasing for all $x \in \mathbb{R}$.

Difficulty Level: B

6. Sketch the graph of the following functions:

(a) $y = 2x - 1$

(b) $y = x^2 - 4x + 3$

(c) $y = -x^2 + 4x - 3$

(d) $y = 2^{-x}$

(e) $y = 3^x$

(f) $y = \log_3 x$

(g) $y = \log \frac{1}{2}x$

(h) $y = \sin x$ for $-2\pi \leq x \leq 2\pi$

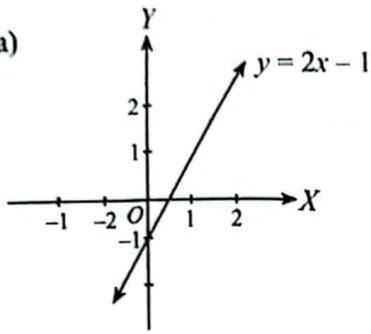
(i) $y = \cos x$ for $-2\pi \leq x \leq 2\pi$

(j) $y = \tan x$ for $-2\pi \leq x \leq 2\pi$

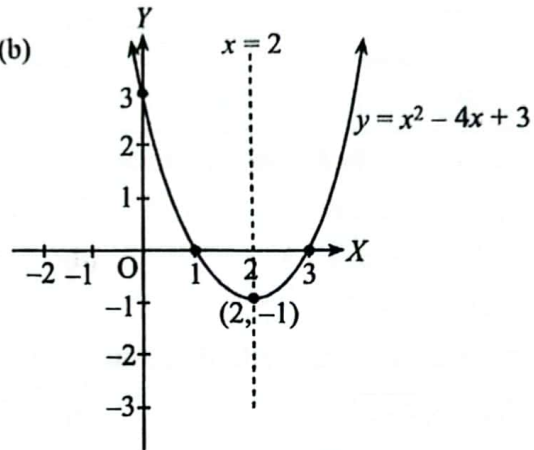
ANSWERS

- | | | | | | |
|----|---------------------------------|---------------------------------|---------------------------------|----------------------------|---------|
| 1. | (a) Odd | (b) Neither | (c) Even | (d) Neither | (e) Odd |
| 2. | (a) Symmetric about y-axis | (b) Symmetric about y-axis | (c) Symmetric about origin | (d) Symmetric about origin | |
| | (e) None | | | | |
| 3. | (a) Periodic with period π | (b) Periodic with period 2π | (c) Periodic with period 2π | (d) Periodic with period 2 | |
| | (e) Periodic with period 4π | | | | |

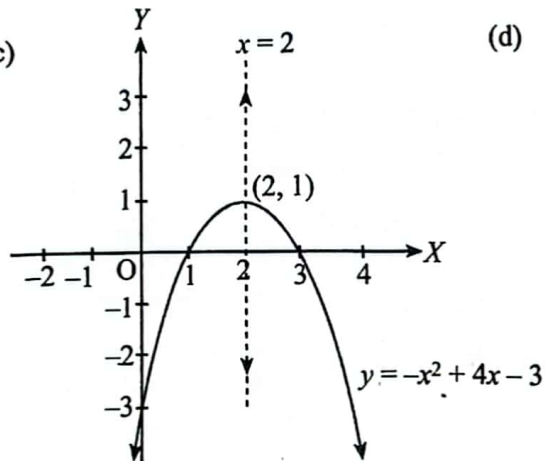
6. (a)



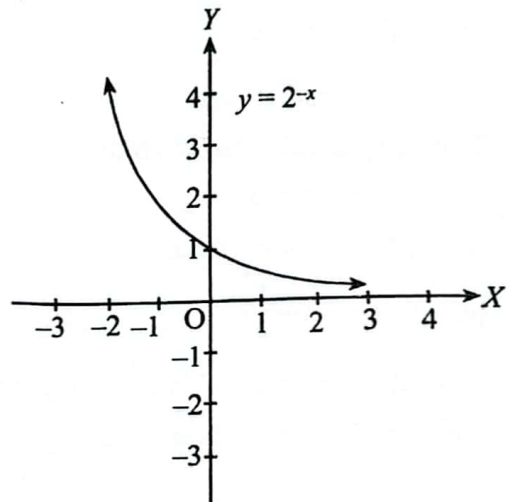
(b)



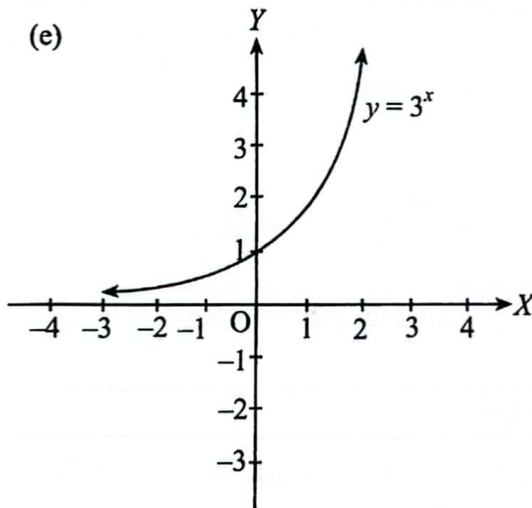
(c)



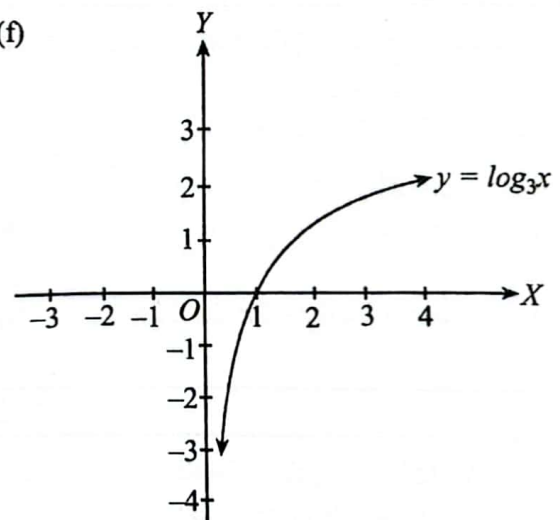
(d)



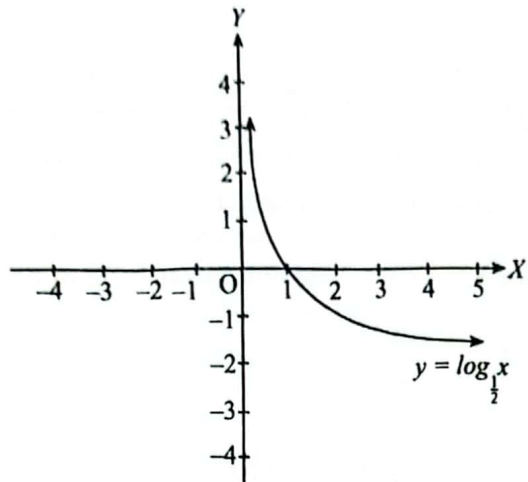
(e)



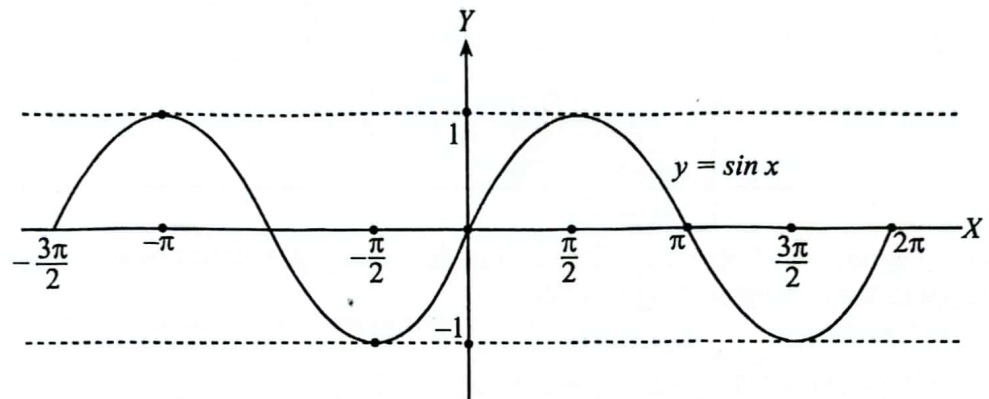
(f)



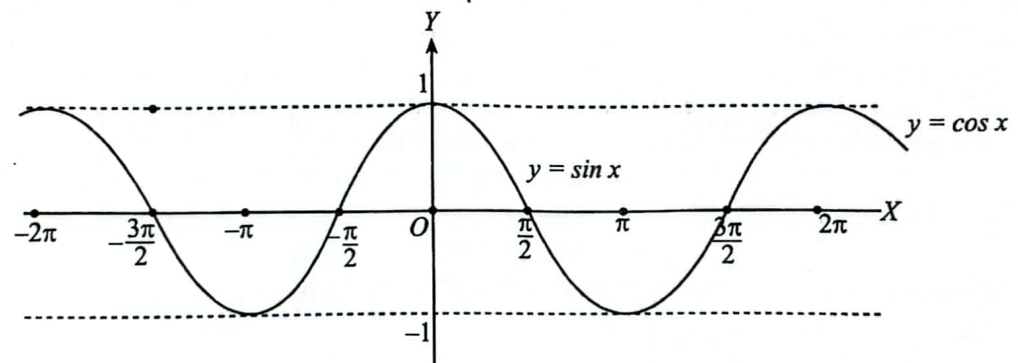
(g)



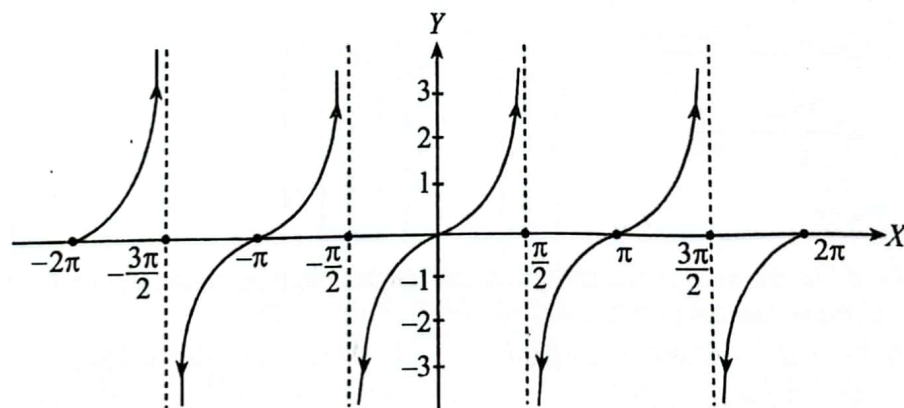
(h)



(i)



(j)

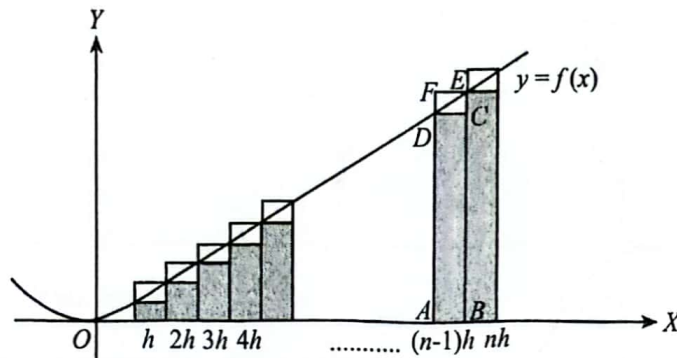


3.5 Application of Anti-derivatives

Area under a Curve Using the Limit of Sum (or Method of Summation)

Let $y = f(x) = x^2$ be a function. We find the area of the plane region bounded by the curve given by $y = x^2$, x -axis and the lines of ordinates $x = a$ and $x = b$. Let us divide the interval $[0, b]$ into n equal sub-intervals having width h . That is, $b - 0 = hn \Rightarrow nh = b$ is the second ordinate and that of $x = a = 0$ is the first ordinate.

Since $y = f(x) = x^2$ so, $y = f(h) = h^2$ as $x = h$.



Let A_n be the sum of the series of the area of the rectangles inscribed by the curve $y = x^2$ (i.e. the sum of area of all the rectangles of type $ABCD$).

Then,

$$A_n = 0.h + h.(h)^2 + h.(2h)^2 + h.(3h)^2 + \dots + h[(n-1)h]^2$$

$$A_n = 0 + h^3 + 4h^3 + 9h^3 + \dots + (n-1)^2 h^3$$

or, $A_n = h^3 [1^2 + 2^2 + 3^2 + \dots + (n-1)^2]$

$$= h^3 \left[\frac{n(n+1)(2n+1)}{6} \right] \quad \left[\because \sum n^2 = \frac{n(n+1)(2n+1)}{6} \right]$$

or, $A_n = \frac{nh(nh+h)(2nh+h)}{6}$

Since $n \rightarrow \infty$, $h \rightarrow 0$ so taking limit at $n \rightarrow \infty$, then we have

$$\lim_{n \rightarrow \infty} A_n = \lim_{h \rightarrow 0} \frac{nh(nh+h)(2nh+h)}{6}$$

$$= \frac{b(b+0)(2b+0)}{6} \quad [\because nh = b]$$

$$= \frac{b^2(2b)}{6} = \frac{2b^3}{6}$$

$$\therefore \lim_{n \rightarrow \infty} A_n = \frac{b^3}{3} \quad (1)$$

Again, let A_n' be the sum of the area of the rectangles circumscribing by the curve $y = x^2$ (i.e. the sum of the area of the rectangles representing by $ABEF$). Then,

$$A_n' = h.(h)^2 + h.(2h)^2 + h.(3h)^2 + \dots + h.(nh)^2 \quad [\because \text{Area} = \text{breadth} \times \text{height}]$$

or, $A_n' = h^3 + 4h^3 + 9h^3 + \dots + n^2 h^3$

$$= h^3 [1^2 + 2^2 + 3^2 + \dots + n^2]$$

$$= h^3 \left[\frac{n(n+1)(2n+1)}{6} \right]$$

Since, $n \rightarrow \infty$, $h \rightarrow 0$ so taking limit as $n \rightarrow \infty$ we have then

$$\begin{aligned} \lim_{n \rightarrow \infty} A_n' &= \lim_{h \rightarrow 0} \frac{nh(nh+h)(2nh+h)}{6} \\ &= \frac{b(b+0)(2b+0)}{6} \quad [\because nh = b] \end{aligned}$$

$$\text{or, } \lim_{n \rightarrow \infty} A_n' = \frac{b^2(2b)}{6} = \frac{b^3}{6} [2] = \frac{b^3}{3}$$

$$\therefore \lim_{n \rightarrow \infty} A_n' = \frac{b^3}{3} \quad (2)$$

Now, let us denote the area of the plane region bounded by the curve $y = x^2$, x -axis and the ordinates $x = a = 0$ and $x = b$ by A . Thus from (1) and (2) we have the relation as

$$\lim_{n \rightarrow \infty} A_n \leq A \leq \lim_{n \rightarrow \infty} A_n'$$

$$\text{or, } \frac{b^3}{3} \leq A \leq \frac{b^3}{3}$$

$$\therefore A = \frac{b^3}{3}.$$

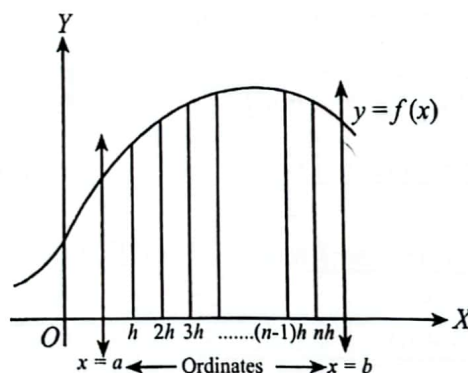
Hence, the area of the plane region under the curve $y = f(x) = x^2$ from $x = 0$ to $x = b$ is $PA = \frac{1}{3} b^3$ which is obtained by the method of the limit of sum.

Alternatively,

To find the area of the plane region bounded above by the curve given by $y = f(x)$, bounded left and right by the ordinates $x = a$ and $x = b$ respectively and bounded below by the x -axis then, by the method of limit of sum we have to divide the horizontal length of the interval $[a, b]$ into n sub-intervals each equal to h . Thus, the plane region under the curve given by $y = f(x)$, divided into n infinitely tiny strips each nearly representing a rectangle. That is $b - a = nh$. Now let A be the area of the plane region bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x -axis. Then we have,

Area of the plane region (A) = (Area of the 1st strip) + (Area of the 2nd strip) + (Area of the 3rd strip) + ... + (Area of the n^{th} strip).

As n increases sufficiently large then $h \rightarrow 0$, so in the limiting position each strip approximately takes the form of a rectangle, then



$$\text{Area of the plane region (A)} = \int_a^b f(x) dx$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} [h \cdot f(h) + hf(2h) + hf(3h) + \dots + hf(nh)] \\
 &= \lim_{h \rightarrow 0} h [f(h) + f(2h) + f(3h) + \dots + f(nh)] \text{ where } nh = b - a
 \end{aligned}$$

Hence, by the limit of sum, the area of the plane region under a curve is

$$A = \int_a^b f(x) dx = \lim_{h \rightarrow 0} h [f(h) + f(2h) + f(3h) + \dots + f(nh)], \text{ where } nh = b - a.$$

Some relation which are useful to find the area by the method of summation

- (i) $\Sigma n = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$
- (ii) $\Sigma n^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$
- (iii) $\Sigma n^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$
- (iv) $S_n = \frac{a(r^n - 1)}{r - 1}$
- (v) $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$

Example: 1

Use the method of summation (or limit of sum) to find the area bounded by the curve $y = x^2$ the x -axis and the ordinates $x=0$ and $x=1$.

Solution:

Here, $y = f(x) = x^2$ be a given curve and the ordinates are $x=0$ and $x=1$. Now, the area of the plane region bounded by the curve $y=x^2$, the x -axis and the ordinates $x=0$ and $x=1$, by using the limit of sum we have,

$$\begin{aligned}
 \text{Required area (A)} &= \int_0^1 f(x) dx \\
 &= \lim_{h \rightarrow 0} h [f(h) + f(2h) + f(3h) + \dots + f(nh)] \text{ where } nh = 1 - 0 = 1
 \end{aligned}$$

$$\begin{aligned}
 \text{or, } A &= \lim_{h \rightarrow 0} h [h^2 + (2h)^2 + (3h)^2 + \dots + (nh)^2] \\
 &= \lim_{h \rightarrow 0} h^3 [1^2 + 2^2 + 3^2 + \dots + n^2]
 \end{aligned}$$

$$\begin{aligned}
 \text{or, } A &= \lim_{h \rightarrow 0} h^3 \left[\frac{n(n+1)(2n+1)}{6} \right] \\
 &= \lim_{h \rightarrow 0} \frac{nh(nh+h)(2nh+h)}{6}
 \end{aligned}$$

$$\text{or, } A = \frac{1(1+0)(2 \times 1 + 0)}{6} = \frac{1 \times 2}{6} = \frac{2}{6} = \frac{1}{3}$$

$\therefore$ The required area (A) = $\frac{1}{3}$ sq. units

Example: 2

Find the area of the plane region bounded by $y = 1 - x$, $y = 0$, $x = 0$, $x = b$ using the limit of sum.

Solution:

Here, the required area of the plane region bounded by the curve $y = f(x) = 1 - x$, $y = 0$ (i.e., axis of x) and the ordinates $x = 0$ and $x = b$ is given by,

$$\begin{aligned}\text{Required area } (A) &= \int_0^b f(x) dx \\ &= \lim_{h \rightarrow 0} h [f(h) + f(2h) + f(3h) + \dots + f(nh)] \text{ where } nh = b - 0 = b\end{aligned}$$

$$\begin{aligned}\text{or, } A &= \lim_{h \rightarrow 0} h [(1 - h) + (1 - 2h) + (1 - 3h) + \dots + (1 - nh)] \\ &= \lim_{h \rightarrow 0} h [1 + 1 + 1 + \dots + 1 \text{ (upto } n^{\text{th}} \text{ term)} - (h + 2h + 3h + \dots + nh)] \\ &= \lim_{h \rightarrow 0} h [n - h(1 + 2 + 3 + \dots + n)] \\ &= \lim_{h \rightarrow 0} \left[nh - h^2 \frac{n(n+1)}{2} \right] \\ &= b - \lim_{h \rightarrow 0} \frac{hn(nh + h)}{2} \quad [\because nh = b] \\ &= b - \frac{b(b+0)}{2} = b - \frac{b^2}{2}\end{aligned}$$

$$\therefore \text{ The required area } (A) = \frac{b(2-b)}{2} \text{ sq. units}$$

Example: 3

Find the area of the plane region bounded by the axis of x , the curve $y = e^{ax}$ and the ordinates $x = 0$, $x = a$ using the limit of sum.

Solution:

Here, $y = f(x) = e^{ax}$

Now, the area of the plane region bounded by the curve $y = e^{ax}$, x -axis and the ordinates $x = 0$, $x = a$ by the limit of sum is given by

$$\begin{aligned}\text{Required area } (A) &= \int_0^a f(x) dx \\ &= \lim_{h \rightarrow 0} h [f(h) + f(2h) + \dots + f(nh)], \text{ where } nh = a\end{aligned}$$

$$\begin{aligned}\text{or, } A &= \lim_{h \rightarrow 0} h [e^{ah} + e^{2ah} + \dots + e^{a.nh}] \\ &= \lim_{h \rightarrow 0} h [e^{1.ah} + e^{2.ah} + e^{3.ah} + \dots + e^{n.ah}] \\ &= \lim_{h \rightarrow 0} h \left[\frac{e^{ah} \{(e^{ah})^n - 1\}}{e^{ah} - 1} \right] \quad \left[\because S_n = \frac{a(r^n - 1)}{r - 1} \right]\end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \left[\frac{e^{ah}(e^{ah} - 1)}{\frac{e^{ah} - 1}{ah} \times a} \right] \\
&= \left[\frac{e^0 [e^{a(a)} - 1]}{\lim_{h \rightarrow 0} \frac{e^{ah} - 1}{ah} \times a} \right] \\
&= \frac{e^0 (e^{a^2} - 1)}{1 \times a} \\
&= \frac{1(e^{a^2} - 1)}{a}
\end{aligned}
\quad \left[\because \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1 \right]$$

$\therefore$ Required area $(A) = \frac{1}{a} (e^{a^2} - 1)$ sq. units.

3.6 Application of Fundamental Theorem

Let $y = f(x)$ be a continuous function defined in an interval $[a, b]$. Here we have to find the area of the plane region between two fixed lines $x = a$ and $x = b = c + h$ under the curve given by $y = f(x)$. For that let us take $x = c$ be a variable line between $x = a$ and $x = b$ as shown in the figure. Now, at first, we find the area between $x = a$ and the variable line $x = c$. Suppose the area be $A(x)$ and also the area between $x = a$ and $x = b$ is $A(b)$.

Obviously, the area between $x = c$ and $x = c + h$ is

$$A(c + h) - A(c) \quad (1)$$

Again, the area between these lines is approximately

$$h \cdot f(x) \quad (2)$$

But the area in (1) and (2) are equal to the area of the plane region between the same lines. So we have,

$$A(c + h) - A(c) = h \cdot f(x)$$

$$\text{or, } \frac{A(c + h) - A(c)}{h} = \frac{h \cdot f(x)}{h}$$

$$\text{or, } A'(c) = f(x) \text{ for } h \rightarrow 0$$

Hence, $A(c)$ is the ant-derivative of $f(x)$. But $f(x)$ has a number of antiderivatives which are different only by a constant. Now, if $F(x)$ is an antiderivative of $f(x)$ then we have by Riemann sum that

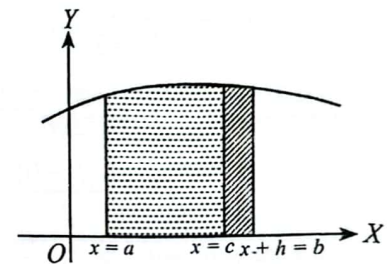
$$A(c) = \int_a^c f(x) dx = F(x) + c$$

$$\text{If } c = a, \text{ then } A(a) = 0 = F(a) + c \Rightarrow c = -F(a)$$

$$\text{And if } c = b \text{ then } A(b) = \int_a^b f(x) dx = F(b) + c$$

$$\Rightarrow A(b) = \int_a^b f(x) dx = F(b) - F(a) \quad [\because c = -F(a)]$$

$$\text{Hence, } A(b) = \int_a^b f(x) dx = F(b) - F(a)$$

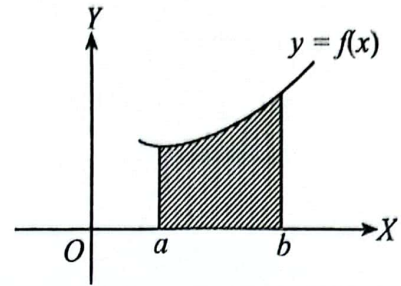


Area as a Definite Integral

To find the area under the curve $y = f(x)$ above x-axis and between the ordinates $x = a$ and $x = b$, we use the definite integral. For this regard, we have following working rule.

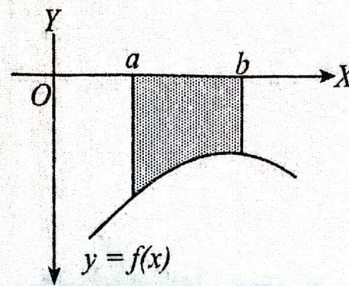
- (a) The area of the region bounded by the continuous curve $y = f(x)$, above x-axis and the lines $x = a$ and $x = b$ is

$$A = \int_a^b f(x) dx$$



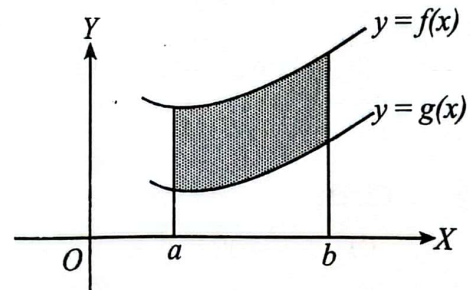
Note: If the bounded region is below x-axis, then the value of area will be negative. In this case we consider magnitude only i.e., we neglect the negative sign.

$$A = \left| \int_a^b f(x) dx \right|$$

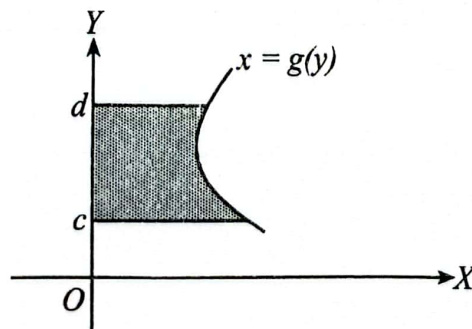


- (b) The area of the region bounded by two continuous curve $y = f(x)$ and $y = g(x)$ and between the ordinates $x = a$ and $x = b$ is

$$A = \left| \int_a^b [f(x) - g(x)] dx \right|$$

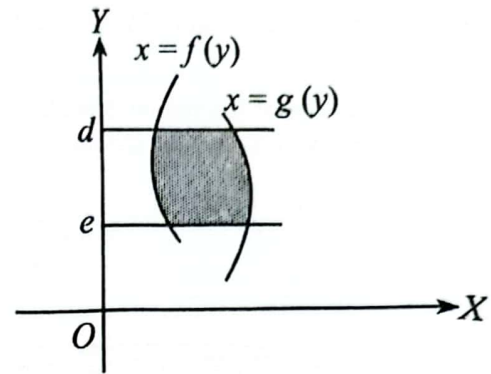


- (c) The area of the region bounded by a continuous curve $x = g(y)$, y-axis and between the lines $y = c$ and $y = d$ is given by $A = \int_c^d g(y) dy$



- (d) The area of the region bounded by two continuous curve $x = f(y)$ and $x = g(y)$ and the lines $y = c$ and $y = d$ is given by

$$A = \int_c^d [g(y) - f(y)] dy$$

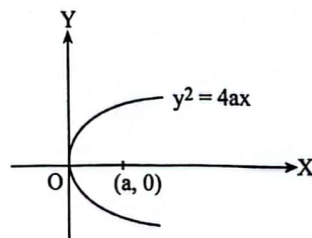


Graph of Some Well-Know Curves

Here we will discuss about the graph of some familiar curves and their chief characteristics, which are given below:

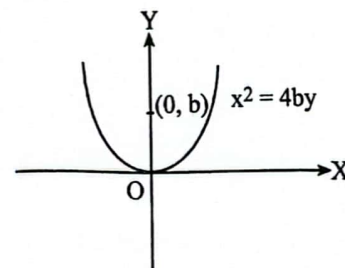
(a) Parabola

(i) $y^2 = 4ax$



Vertex = (0, 0)
Focus = (a, 0)

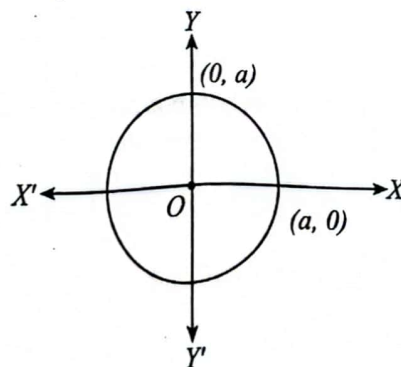
(ii) $x^2 = 4by$



Vertex = (0, 0)
Focus = (0, b)

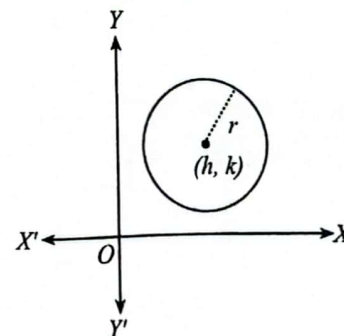
(b) Circle

(i) $x^2 + y^2 = a^2$



Centre = (0, 0)
Radius = a

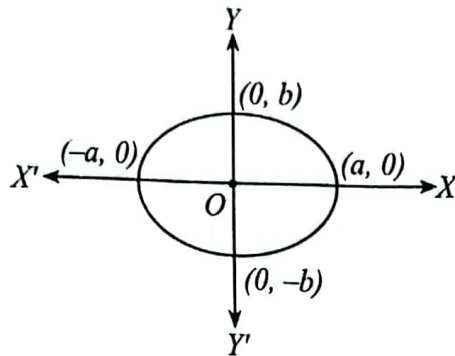
(ii) $(x - h)^2 + (y + k)^2 = r^2$



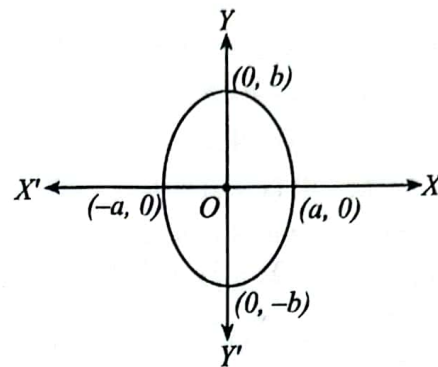
Centre = (h, k)
Radius = r

(c) Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(i) When $a > b$



(ii) When $a < b$



Example: 1

Find the area bounded by curve $y = 3x^2$, x-axis and the ordinates $x=1, x=2$

Solution:

The required area is $A = \int_a^b y dx$

Where, $y = 3x^2, a = 1, b = 2$

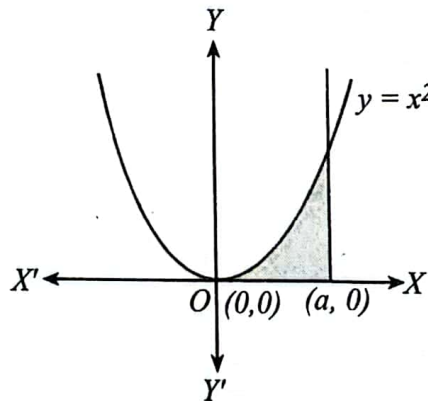
$$\begin{aligned} A &= \int_1^2 3x^2 dx = 3 \cdot \left[\frac{x^3}{3} \right]_1^2 \\ &= [x^3]_1^2 = 8 - 1 = 7 \text{ sq. unit} \end{aligned}$$

Example: 2

Find the area bounded by the x-axis and curve $y=x^2$ and ordinates at the point $(a, 0)$.

Solution:

The graph of this curve $y = x^2$ (parabola) is shown in the following figure.



The required area under the curve $y = x^2$ and between two ordinates $x = 0$ and $x = a$ is

$$\begin{aligned} \int_0^a y dx &= \int_0^a x^2 dx = \left[\frac{x^3}{3} \right]_0^a = \frac{a^3}{3} - \frac{0}{3} \\ &= \frac{a^3}{3} \text{ Sq. unit} \end{aligned}$$

Example: 3

Find the area enclosed by x-axis and the curve $y = x^2 - 10x + 24$.

Solution:

When $y = 0$ then $x^2 - 10x + 24 = 0$

$$\text{or, } x^2 - 6x - 4x + 24 = 0$$

$$\text{or, } x(x - 6) - 4(x - 6) = 0$$

$$\text{or, } (x - 4)(x - 6) = 0$$

or, $x = 4$ or $x = 6$. So, the curve meets x-axis at $(4, 0)$ and $(6, 0)$. Thus the required area under the curve $y = x^2 - 10x + 24$ and ordinates $x = 4$ and $x = 6$. So,

$$\begin{aligned} A &= \int_4^6 y dx = \int_4^6 (x^2 - 10x + 24) dx \\ &= \left[\frac{x^3}{3} - \frac{10x^2}{2} + 24x \right]_4^6 \\ &= \frac{6^3}{3} - \frac{10 \cdot 6^2}{2} + 24 \cdot 6 - \left[\frac{4^3}{3} - \frac{10 \cdot 4^2}{2} + 24 \cdot 4 \right] \\ &= \frac{216}{3} - 180 + 144 - \frac{64}{3} + 80 - 96 \\ &= \frac{216}{3} - \frac{64}{3} - (180 - 80) + 144 - 96 \\ &= \frac{152}{3} - 100 + 48 \\ &= \frac{152 - 300 + 144}{3} = \frac{-4}{3} = \frac{4}{3} \text{ Sq. unit} \end{aligned}$$

Example: 4

Find the area of the region between $y^2 = 4ax$ and $x^2 = 4ay$.

Solution:

We have given curves $y^2 = 4ax$ and $x^2 = 4ay$

Solving these two equations, we get

$$\left(\frac{x^2}{4a} \right)^2 = 4ax$$

$$\frac{x^4}{16a^2} = 4ax$$

$$\text{or, } x^4 - 64a^3x = 0$$

$$\text{or, } x(x^3 - 64a^3) = 0$$

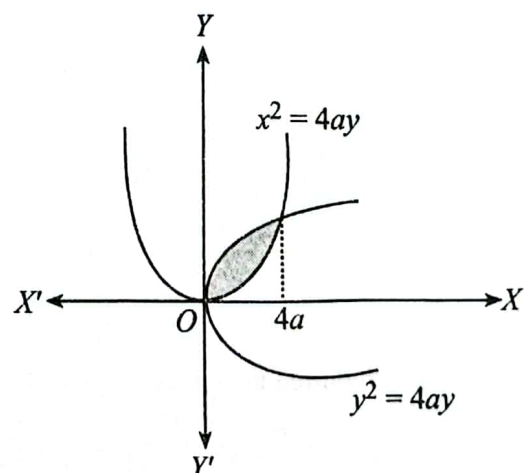
$$\therefore \text{ either } x = 0$$

$$\text{or, } x^3 = 64a^3$$

$$\therefore x = 4a$$

$$\text{From } x^2 = 4ay$$

$$\text{also, } y = \frac{(4a)^2}{4a} = 4a$$



Thus the curves meet at $(0, 0)$ and $(4a, 4a)$. So, the required area between given curves is given by

$$\begin{aligned}
 A &= \int_a^b [g(x) - f(x)] dx \\
 &= \int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a} \right] dx \\
 &= \int_0^{4a} \left[2\sqrt{a}\sqrt{x} - \frac{x^2}{4a} \right] dx \\
 &= \left[2\sqrt{a} \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{4a} \frac{x^3}{3} \right]_0^{4a} \\
 &= \left[\frac{4\sqrt{a}}{3} x^{\frac{3}{2}} - \frac{1}{12a} x^3 \right]_0^{4a} \\
 &= \left(\frac{4\sqrt{a}}{3} (4a)^{\frac{3}{2}} - \frac{1}{12a} (4a)^3 \right) - (0 - 0) \\
 &= \frac{4\sqrt{a}}{3} \times 8a \times \sqrt{a} - \frac{1}{12a} 64a^3 \\
 &= \frac{32a^2}{3} - \frac{16a^2}{3} \\
 &= \frac{16a^2}{3} \text{ sq. unit}
 \end{aligned}$$

Example: 5

Find the area of the circle $x^2 + y^2 = a^2$ by using the method of integration.

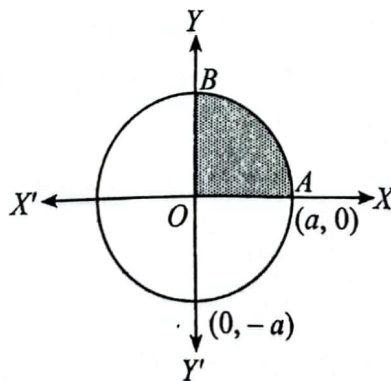
Solution:

We have the curve $x^2 + y^2 = a^2$

When $x = 0$ then $y = \pm a$

When $y = 0$ then $x = \pm a$.

So, the given circle with centre $(0, 0)$ meets x-axis at $(\pm a, 0)$ and y-axis at $(0, \pm a)$ as shown in the figure.



Now, area of $OAB = \int_0^a y dx$

Put $x = a \sin \theta$

On differentiating, we get $dx = a \cos \theta d\theta$

Also, when $x = 0$, $\theta = 0$

when $x = a$, $\theta = \frac{\pi}{2}$

So,

$$\begin{aligned}
 \int_0^a \sqrt{a^2 - x^2} dx &= \int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2 \sin^2 \theta} \cdot a \cos \theta d\theta \\
 &= \int_0^{\frac{\pi}{2}} a^2 \cos \theta \cdot \cos \theta d\theta \\
 &= a^2 \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta \\
 &= a^2 \int_0^{\frac{\pi}{2}} \frac{(1 + \cos 2\theta)d\theta}{2} \\
 &= \frac{a^2}{2} \left[\int_0^{\frac{\pi}{2}} d\theta + \int_0^{\frac{\pi}{2}} \cos 2\theta d\theta \right] \\
 &= \frac{a^2}{2} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}} \\
 &= \frac{a^2}{2} \left[\frac{\pi}{2} + \frac{\sin \frac{2\pi}{2}}{2} - 0 \right] \\
 &= \frac{a^2}{2} \left[\frac{\pi}{2} + 0 \right] = \frac{\pi a^2}{4}
 \end{aligned}$$

So, the required area (A) = 4 area of OAB ,

$$\therefore A = 4 \cdot \frac{\pi a^2}{4} = \pi a^2$$

EXERCISE 3(IV)**Difficulty Level: A**

1. Find the area of the plane region bounded by the x -axis, the following curves and the given ordinates using the limit of sum (or summation method):

(a) $y = x^2, x = a, x = b$	(b) $y = x, x = 0, x = a$
(c) $y = 3x^2, x = 0, x = c$	(d) $y = x^3, x = 0, x = 1$
(e) $y = 4x^2, x = 0, x = a$	(f) $y = e^x, x = 0, x = a$
(g) $y = e^{3x}, x = 0, x = b$	(h) $y = e^{bx}, x = 0, x = b$
2. Find the area bounded by x -axis and the following curves and ordinates:

(a) $y = 3x^2 - 2; x = 1, x = 4$	(b) $y = x; x = 0, x = 2$
(c) $y^2 = 4ax; x = 0, x = a$	
3. Find the area bounded by y -axis and the following curves and abscissa:

(a) $y^2 = 4ax, y = 1, y = 3$	(b) $x^2 = 4by, y = 0, y = b$
(c) $y = x^3; y = 0, y = 4$	

Difficulty Level: B

4. Find the area enclosed by axis of x and the following curves:

(a) $y = 3x - 5x^2$	(b) $y = x^2 - 16$
(c) $y = x^2 - 8x + 15$	(d) $y = x^2 - 10x + 24$
5. Find the area of the region between the following curves:

(a) $y^2 = x$ and $y = x$	(b) $y^2 = 4x$ and $y = x$
(c) $y^2 = 4ax$ and $x = a$	(d) $y^2 = x^3$ and $x = 4$
(e) $y^2 = 4x$ and the line $x = y$	(f) $y^2 = 16x$ and the line $y = 2x$
(g) $y^2 = 4ax$ and the line $x = y$	(h) $y^2 = 16x$ and the line $y = x$
(i) $y = x^2$ and the line $y = 2x$	
(j) $y = x^3$ and the line $y = x$ lying in the first quadrant	
(k) $y = x^2 + 5$ and $y = x^2$ between the ordinates $x = 1$ and $x = 2$.	
6. Find the area of the circle by using integration:

(a) $x^2 + y^2 = 9$	(b) $x^2 + y^2 = 25$
---------------------	----------------------
7. Find the area of ellipse by using integration:

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	(b) $\frac{x^2}{9} + \frac{y^2}{16} = 1$
---	--

ANSWERS

- | | | | | | |
|----|--------------------------|-------------------------------|----------------------------|---------------------|----------------------|
| 1. | (a) $\frac{1}{3}(b-a)^3$ | (b) $\frac{a^2}{2}$ | (c) c^3 | (d) $\frac{1}{4}$ | (e) $\frac{4}{3}a^3$ |
| | (f) $(e^a - 1)$ | (g) $\frac{1}{3}(e^{3b} - 1)$ | (h) $\frac{1}{b}(e^b - 1)$ | | |
| 2. | (a) 57 | (b) 2 | (c) $\frac{4}{3}a^2$ | | |
| 3. | (a) $\frac{13}{6a}$ | (b) $\frac{4}{3}b^2$ | (c) 64 | | |
| 4. | (a) $\frac{63}{50}$ | (b) $\frac{256}{3}$ | (c) $\frac{4}{3}$ | (d) $\frac{4}{3}$ | |
| 5. | (a) $\frac{1}{6}$ | (b) $\frac{8}{3}$ | (c) $\frac{8a^2}{3}$ | (d) $25\frac{3}{5}$ | |
| | (e) $\frac{8}{3}$ | (f) $\frac{16}{3}$ | (g) $\frac{8a^2}{3}$ | (h) $\frac{128}{3}$ | |
| | (i) $\frac{4}{3}$ | (j) $\frac{1}{4}$ | (k) 5 | | |
| 6. | (a) 9π | (b) 25π | | | |
| 7. | (a) πab | (b) 12π | | | |

□□□

Differential Equations

4.1 Introduction

Differential equation is an important branch of mathematics which has wide applications. It plays a vital role in the solution of many problems encountered in modeling all natural phenomena including engineering, physical, biological and social sciences. Many practical problems like decay of certain substance, change in temperature of body, increase in population etc., can be solved with the help of differential equations. In this chapter, we will define and explain some basic concepts of ordinary differential equations; order and degree, solution of differential equation, and different types of differential equations.

4.2 Ordinary Differential Equation

The equation that contains the derivative or differentials (i.e., $\frac{dy}{dx}$) is called differential equation.

A differential equation involving derivatives with respect to a single independent variable is called an ordinary differential equation (ODE). For example, the equations

$$\frac{dy}{dx} = 3x + 1, \quad x \frac{dy}{dx} + \sin x = 5, \quad \frac{d^2y}{dx^2} = \frac{dy}{dx} + xy$$

etc. are ordinary differential equations.

4.3 Order and Degree of Ordinary Differential Equation

The order of highest derivative term appearing in differential equation is called the order of the differential equation. For example,

$\frac{d^2y}{dx^2} - 5 \frac{dy}{dx} = 0$ is a differential equation of second order because the highest derivative occurring in it is 2.

The degree of the differential equation is the degree of highest derivative when differential coefficients are rational and free from fraction. For example,

(a) $x dx + y dy = 0$ is first order and first degree differential equation.

(b) $\frac{d^2y}{dx^2} + 2 \left(\frac{dy}{dx} \right)^3 = xy$ is the second order and first degree differential equation.

(c) $y \frac{dy}{dx} - x \left(\frac{dy}{dx} \right)^2 = C$ is the first order and second degree differential equation.

4.4 Solution of a Differential Equation

Any relation between the dependent and independent variables which is free from derivatives or differential and satisfy the given differential equation is called the solution of the differential equation.

The process of finding the solutions of a differential equation is to take the integration of the differential equation.

Consider the equation $y = ce^x$ (1)

Differentiate with respect to x , $\frac{dy}{dx} = ce^x$

$$\text{i.e. } \frac{dy}{dx} = y$$

$\therefore y = ce^x$ is general solution of the differential equation $\frac{dy}{dx} - y = 0$

Again consider $y = a \cos x + b \sin x$.

Differentiate with respect to x ,

$$\frac{dy}{dx} = -a \sin x + b \cos x$$

Again, differentiate with respect to x ,

$$\begin{aligned} \frac{d^2y}{dx^2} &= -a \cos x - b \sin x \\ &= -(a \cos x + b \sin x) = -y \end{aligned}$$

$\therefore y = a \cos x + b \sin x$ is general solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

Thus the number of arbitrary constants in the general solution is equal to the order of the differential equation and vice versa.

4.5 Types of Differential Equations

The differential equations are of two types

(a) Ordinary differential equations (ODE)

The equation involving the differential coefficient with respect to a single independent variable is called the ordinary differential equation. For example,

$$(i) \quad xy \frac{dy}{dx} + y^2 = 3 \quad (ii) \quad \frac{dy}{dx} = 0 \quad (iii) \quad \frac{d^2y}{dx^2} + 5 \frac{dy}{dx} + 3 = x \text{ etc.}$$

are the ordinary differential equations.

(b) Partial differential equations (PDE)

The differential equation involving partial derivatives with respect to more than one independent variables is a partial differential equation. For examples,

$$(i) \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} \quad (ii) \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \quad (iii) \quad \left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial z}{\partial y} = 0 \text{ etc.}$$

4.6 Differential Equation of First Order and First Degree

There are two standard forms of differential equations of first order and first degree. They are

$$(a) \quad \frac{dy}{dx} = f(x, y)$$

$$(b) \quad M(x, y) dx + N(x, y) dy = 0$$

We now discuss various techniques to solve such differential equations.

4.7 Separation of Variables

If a differential equation of the first order and first degree is of the form

$$f_1(x) dx = f_2(y) dy \quad (1)$$

where $f_1(x)$ is a function of x only and $f_2(y)$ is a function of y only, then we say that the variables are separable in the given differential equation.

Such equations are solved by integrating both sides of (1) and adding an arbitrary constant of integration to any one of the two sides. Thus, solution of (1) is

$$\int f_1(x) dx = \int f_2(y) dy + C$$

Note: The constant of integration C can be selected in any suitable form. For example, C can be replaced by $\frac{C}{2}$, $\log C$, $\tan C$, $\tan^{-1} C$, e^C etc.

Example: 1

Solve: $x^2 \frac{dy}{dx} = 2$

Solution:

We have, $x^2 \frac{dy}{dx} = 2$

or, $\frac{dy}{dx} = \frac{2}{x^2}$

or, $dy = \frac{2}{x^2} dx$

Integrating both sides

$$\int dy = 2 \int \frac{1}{x^2} dx$$

or, $y = 2 \cdot \frac{x^{-2+1}}{-2+1} + C$

$\therefore y = \frac{-2}{x} + C$

Example: 2

Solve: $\frac{dy}{dx} = x \log x$

Solution:

We have, $\frac{dy}{dx} = x \log x$

or, $dy = x \log x dx$

Integrating both sides, we get

$$\int dy = \int x \log x dx$$

or, $y = \log x \int x dx - \int \left[\frac{d}{dx} (\log x) \int x dx \right] dx$

$$\text{or, } y = \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$$

$$\text{or, } y = \log x \cdot \frac{x^2}{2} - \frac{1}{2} \int x dx$$

$$\text{or, } y = \log x \cdot \frac{x^2}{2} - \frac{1}{2} \cdot \frac{x^2}{2} + C$$

$$\therefore y = \log x \cdot \frac{x^2}{2} - \frac{1}{4} x^2 + C$$

Example: 3

$$\text{Solve: } (1 + \cos x) dy = (1 - \cos x) dx$$

Solution:

$$\text{We have, } (1 + \cos x) dy = (1 - \cos x) dx$$

$$\text{or, } dy = \frac{1 - \left(1 - 2 \sin^2 \frac{x}{2}\right)}{1 + \left(2 \cos^2 \frac{x}{2} - 1\right)} dx$$

$$= \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}$$

$$\text{or, } dy = \tan^2 \frac{x}{2} dx$$

Integrating both sides,

$$\int dy = \int \tan^2 \frac{x}{2} dx$$

$$\text{or, } y = \int \left(\sec^2 \frac{x}{2} - 1 \right) dx$$

$$\text{or, } y = \frac{\tan \frac{x}{2}}{\frac{1}{2}} - x + C$$

$$\therefore y = 2 \tan \frac{x}{2} - x + C$$

Example: 4

$$\text{Solve: } (1 + x^2) dy = (1 + y^2) dx$$

Solution:

$$\text{We have, } (1 + x^2) dy = (1 + y^2) dx$$

$$\text{or, } (1 + x^2) dy - (1 + y^2) dx = 0$$

$$\text{or, } \frac{dy}{1 + y^2} - \frac{dx}{1 + x^2} = 0$$

On integration,

$$\tan^{-1} y - \tan^{-1} x = \tan^{-1} C$$

$$\text{or, } \tan^{-1} \left(\frac{y-x}{1+xy} \right) = \tan^{-1} C$$

$$\therefore \frac{y-x}{1+xy} = C$$

Example: 5

$$\text{Solve: } \sec^2 x \tan y \, dy + \sec^2 y \tan x \, dx = 0$$

Solution:

$$\text{We have, } \sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$$

Dividing both sides by $\tan x \tan y$

$$\frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0$$

On integration,

$$\log \tan x + \log \tan y = \log C$$

$$\log (\tan x \tan y) = \log C$$

$$\therefore \tan x \tan y = C$$

Example: 6

$$\text{Solve: } (xy^2 + x) \, dx + (yx^2 + y) \, dy = 0$$

Solution:

$$\text{We have, } (xy^2 + x) \, dx + (yx^2 + y) \, dy = 0$$

$$\text{or, } x(y^2 + 1) \, dx + y(x^2 + 1) \, dy = 0$$

Dividing both sides by $(x^2 + 1)(y^2 + 1)$, we get

$$\frac{x}{x^2 + 1} dx + \frac{y}{y^2 + 1} dy = 0$$

$$\frac{2x}{2(x^2 + 1)} dx + \frac{2y}{2(y^2 + 1)} dy = 0$$

On integration,

$$\frac{1}{2} \log (x^2 + 1) + \frac{1}{2} \log (y^2 + 1) = \frac{1}{2} \log C$$

$$\text{or, } \log [(x^2 + 1)(y^2 + 1)] = \log C$$

$$\therefore (x^2 + 1)(y^2 + 1) = C$$

$$\text{Equation of the form } \frac{dy}{dx} = f(ax + by + c)$$

On solving the differential equation of the form $\frac{dy}{dx} = f(ax + by + c)$ we put $ax + by + c = v$ which makes easier to separate the variables. One example is done as below.

Example: 7

$$\text{Solve: } (x + y + 1) \frac{dy}{dx} = 1$$

[CTEVT 2071]**Solution:**

Here, the given equation is

$$(x + y + 1) \frac{dy}{dx} = 1 \quad (1)$$

Let $x + y + 1 = v$

Differentiating with respect to x on both sides,

$$1 + \frac{dy}{dx} + 0 = \frac{dv}{dx}$$

$$\text{or, } \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Now, equation (1) becomes as,

$$v \left(\frac{dv}{dx} - 1 \right) = 1$$

$$\text{or, } \frac{dv}{dx} - 1 = \frac{1}{v}$$

$$\text{or, } \frac{dv}{dx} = 1 + \frac{1}{v} = \frac{v+1}{v}$$

$$\text{or, } v \frac{dv}{dx} = v + 1$$

$$\text{or, } \frac{v dv}{1+v} = dx$$

$$\text{or, } \frac{[(1+v)-1]}{1+v} dv = dx$$

$$\text{or, } \left[1 - \frac{1}{1+v} \right] dv = dx$$

On integrating,

$$\int dv - \int \frac{1}{1+v} dv = \int dx$$

$$\text{or, } v - \log(1+v) = x + C_1$$

$$\text{or, } x + y + 1 - \log(1 + x + y + 1) = x + C_1$$

$$\text{or, } y + 1 - \log(x + y + 2) = C_1$$

$$\text{or, } y - \log(x + y + 2) = C_1 - 1$$

$$\therefore y - \log(x + y + 2) = C, \text{ where } C = C_1 - 1$$

EXERCISE - 4(I)

Difficulty Level: A

Solve the following differential equations:

1. $x \frac{dy}{dx} = y$

2. $\frac{dy}{dx} = -\frac{y}{x}$

3. $(x^2 + 1) \frac{dy}{dx} = xy$

4. $\frac{dy}{dx} = \frac{e^x + 1}{y}$

5. $\frac{dy}{dx} + 2x = e^{3x}$

7. $\frac{dy}{dx} = \frac{1 + \cos 2y}{1 - \cos 2x}$

9. $(1+x)y dx + (1+y)x dy = 0$

6. $\frac{dy}{dx} = (e^x + 1)y$

8. $\sqrt{1-x^2} dy + \sqrt{1-y^2} dx = 0$

[CTEVT 2073]

Difficulty Level: B

10. $\log\left(\frac{dy}{dx}\right) = ax + by$

12. $(xy+x) dy - (xy+y) dx = 0$

14. $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

16. $\frac{dy}{dx} = (x+y+1)^2$ [CTEVT 2073]

18. $\frac{dy}{dx} = \frac{x+y}{x+y+1}$

11. $e^{x-y} dx + e^{y-x} dy = 0$

13. $\tan x dy + \tan y dx = 0$

15. $x^2 dy - y^2 dx = 0$

17. $1 + \frac{dy}{dx} = e^{x+y}$

19. $\cos y dx + (1 + 2e^{-x}) \sin y dy = 0$ given $y = \frac{\pi}{4}$ when $x = 0$.

ANSWERS

1. $y = Cx$

3. $y = C\sqrt{x^2+1}$

5. $y = \frac{1}{3}e^{3x} - x^2 + C$

7. $\cot x + \tan y = C$

9. $\log xy + x + y = C$

11. $e^{2x} + e^{2y} = c$

13. $\sin x \sin y = C$

15. $x - y = cxy$

17. $x + e^{-(x+y)} = C$

19. $(e^x + 2) \sec y = 3\sqrt{2}$

2. $xy = C$

4. $y^2 = 2e^x + 2x + c$ where $c = c_1$

6. $\log y = e^x + x + C$

8. $x\sqrt{1-y^2} + y\sqrt{1-x^2} = C$

10. $be^{ax} + ae^{-by} = C$

12. $x = cye^{y-x}$

14. $e^y = e^x + \frac{x^3}{3} + C$

16. $x + y + 1 = \tan(x + C)$

18. $y + \frac{1}{2} \log(2x + 2y + 1) = x + C$

□□□

4.8 Homogeneous Differential Equations

A differential equation of first order and first degree of the form $\frac{dy}{dx} = \frac{h(x, y)}{g(x, y)}$ is said to be homogenous if both the functions $h(x, y)$ and $g(x, y)$ are homogeneous functions of same degree. The homogenous equation can be expressed as

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

The homogenous differential equation can always be transformed into a variable separable form by setting $y = vx$ to define new variable.

$$\text{We have } \frac{dy}{dx} = \frac{h(x, y)}{g(x, y)} = f\left(\frac{y}{x}\right) \quad (1)$$

Put $y = vx$, and differentiate with respect to x

$$\text{We have, } \frac{dy}{dx} = v + x \frac{dv}{dx} \quad (2)$$

From (1) and (2), we get

$$v + x \frac{dv}{dx} = f(v)$$

$$\text{i.e., } \frac{dv}{dx} = \frac{f(v) - v}{x}$$

Separating the variables, we get $\frac{dx}{x} = \frac{dv}{f(v) - v}$

Integrating, we get

$$\log x + C = \int \frac{dv}{f(v) - v}$$

After integration, replace v by $\frac{y}{x}$ to get the solution of the given differential equation.

Example: 1

$$\text{Solve: } \frac{dy}{dx} = \frac{x + y}{x}$$

Solution:

$$\text{We have, } \frac{dy}{dx} = \frac{x + y}{x}$$

$$\text{i.e., } \frac{dy}{dx} = 1 + \frac{y}{x} \quad (1)$$

It is homogenous equation, so we put $y = vx$ and differentiating we get

$$\therefore \frac{dy}{dx} = v + x \frac{dv}{dx} \quad (2)$$

From (1) and (2), we get

$$v + x \frac{dv}{dx} = 1 + v$$

$$\text{or, } x \frac{dv}{dx} = 1$$

or, $dv = \frac{dx}{x}$ [$\because$ Separating the variables]

On integration, we get

$$v = \log x + \log C$$

or, $v = \log(Cx)$

Replacing $v = \frac{y}{x}$

$$\therefore y = x \log(Cx)$$

Example: 2

Solve: $\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$

Solution:

We have, $\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$

$$\frac{dy}{dx} = \frac{1}{2} + \frac{1}{2} \left(\frac{y}{x}\right)^2 \quad (1)$$

It is homogenous equation, so we put $y = vx$ and differentiating we get

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \quad (2)$$

From (1) and (2), we get

$$v + x \frac{dv}{dx} = \frac{1}{2} + \frac{1}{2} v^2$$

or, $x \frac{dv}{dx} = \frac{1}{2} + \frac{1}{2} v^2 - v = \frac{v^2 - 2v + 1}{2}$

or, $\frac{2dv}{v^2 - 2v + 1} = \frac{dx}{x}$

or, $\frac{2dv}{(v-1)^2} = \frac{dx}{x}$

Integrating, $-\frac{2}{(v-1)} = \log x + \log C$

or, $-\frac{2}{\left(\frac{y}{x} - 1\right)} = \log(Cx)$

or, $-\frac{2x}{(y-x)} = \log Cx.$

$\therefore 2x = (x-y) \log Cx.$

Example: 3

Solve: $x^2 \frac{dy}{dx} = x^2 + xy + y^2$

Solution:

We have, $x^2 \frac{dy}{dx} = x^2 + xy + y^2$

$$\text{or, } \frac{dy}{dx} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2 \quad (1)$$

It is homogeneous equation. So we put $y = vx$ and differentiating we get,

$$\therefore \frac{dy}{dx} = v + x \frac{dv}{dx} \quad (2)$$

From (1) and (2), we get

$$v + x \frac{dv}{dx} = 1 + v + v^2$$

$$\text{or, } x \frac{dv}{dx} = 1 + v^2$$

$$\text{or, } \frac{dv}{1 + v^2} = \frac{dx}{x} \quad [\text{Separating variables}]$$

On integration,

$$\tan^{-1} v = \log x + \log C \quad \left[\because \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C \right]$$

$$\therefore \tan^{-1} \left(\frac{y}{x} \right) = \log (Cx)$$

Example: 4

$$\text{Solve: } 2 \frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$$

Solution:

$$\text{We have, } 2 \frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2} \quad (1)$$

It is homogeneous equation, so we put $y = vx$ and on differentiating we get,

$$\therefore \frac{dy}{dx} = v + x \frac{dv}{dx} \quad (2)$$

From (1) and (2), we get

$$2 \left(v + x \frac{dv}{dx} \right) = v + v^2$$

$$\text{or, } 2v + 2x \frac{dv}{dx} = v + v^2$$

$$\text{or, } 2x \frac{dv}{dx} = v^2 - v$$

$$\text{or, } \frac{2dv}{v^2 - v} = \frac{dx}{x}$$

$$\text{or, } \frac{2dv}{v(v-1)} = \frac{dx}{x}$$

$$\text{or, } 2 \left(\frac{1}{v-1} - \frac{1}{v} \right) dv = \frac{dx}{x}$$

On integrating,

$$2[\log(v-1) - \log v] = \log x + \log C$$

$$\text{or, } 2 \log \left(\frac{y-1}{y} \right) = \log (Cx)$$

$$\text{or, } \log \left(1 - \frac{1}{y} \right)^2 = \log (Cx)$$

$$\text{or, } \log \left(\frac{y-x}{y} \right)^2 = \log Cx$$

$$\text{or, } \left(\frac{y-x}{y} \right)^2 = Cx$$

$$\therefore (y-x)^2 = Cxy^2$$

Example: 5

$$\text{Solve: } x \frac{dy}{dx} = y - x \tan \left(\frac{y}{x} \right)$$

Solution:

The given equation can be written as

$$\frac{dy}{dx} = \frac{y}{x} - \tan \left(\frac{y}{x} \right) \quad (1)$$

Which being a function of $\left(\frac{y}{x} \right)$, is homogeneous. So we put $y = vx$ and on differentiating we get,

$$\therefore \frac{dy}{dx} = v + x \frac{dv}{dx} \quad (2)$$

From (1) and (2), we get

$$v + x \frac{dv}{dx} = v - \tan v$$

$$\text{or, } x \frac{dv}{dx} = -\tan v = -\frac{1}{\cot v}$$

$$\text{or, } \cot v \, dv = -\frac{dx}{x} \quad [\text{Separating variables}]$$

$$\text{Integrating, } \int \frac{\cos v}{\sin v} \, dv = - \int \frac{1}{x} \, dx$$

$$\text{or, } \log (\sin v) = -\log (x) + \log C$$

$$\text{or, } \log (\sin v) + \log x = \log C$$

$$\text{or, } \log |x \sin v| = \log C$$

$$\text{or, } x \sin v = C$$

$$\therefore x \sin \left(\frac{y}{x} \right) = C$$

EXERCISE - 4(II)**Difficulty Level: B**

Solve the following different equations.

1. $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$

2. $\frac{dy}{dx} = \frac{y - 2x}{x}$

3. $(x^2 - y^2) dy = 2xy dx$

4. $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$

5. $\frac{dy}{dx} = \frac{y}{x} + \sin\left(\frac{y}{x}\right)$

6. $x dy - y dx = \sqrt{x^2 + y^2} dx$

7. $y^2 dx + (x^2 - xy + y^2) dy = 0$

8. $\frac{dy}{dx} = \frac{y}{x} - \sin^2 \frac{y}{x}$

9. $x^2 y dx - (x^3 + y^3) dy = 0$

10. $x \sin \frac{y}{x} dy = \left(y \sin \frac{y}{x} - x\right) dx$

ANSWERS

1. $x^2 + y^2 = Cx$

2. $x + 2x \log x = Cx$

3. $y = C(x^2 + y^2)$

4. $\tan\left(\frac{y}{x}\right) = Cx$

5. $\tan\left(\frac{y}{2x}\right) = Cx$

6. $y + \sqrt{x^2 + y^2} = Cx^2$

7. $y = C \tan^{-1}\left(\frac{y}{x}\right)$

8. $\cot\left(\frac{y}{x}\right) = \log x + C$

9. $y = ce^{\frac{x^3}{3y^3}}$

10. $\cos \frac{y}{x} = \ln x + C.$

□□□

4.9 Linear Differential Equation

A differential equation is said to be "Linear Differential Equation" of first order if it is in the form

$$\frac{dy}{dx} + Py = Q \quad (1)$$

where, P and Q are the functions of x only (but not of y) or constants.

Note: Another possible form of linear differential equation is $\frac{dx}{dy} + Px = Q$, where P and Q are functions of y only (but not of x) or constants.

Solving Method:

To solve such type of differential equations, we first determine a function R(say) of x which is called "Integrating factor (I.F.)". Then we multiply both sides of the given equation by R as follows:

$$R \frac{dy}{dx} + PRy = RQ \quad (2)$$

Here, we choose R such that L.H.S. of (2) becomes differential coefficient (i.e. derivatives) of (yR).

$$\text{That is, } \frac{d}{dx}(yR) = R \frac{dy}{dx} + PRy$$

$$\text{or, } R \frac{dy}{dx} + y \frac{dR}{dx} = R \frac{dy}{dx} + PRy$$

$$\Rightarrow y \frac{dR}{dx} = PRy$$

$$\text{or, } \frac{dR}{R} = P dx$$

On integration, we get

$$\text{or, } \log R = \int P dx$$

$\therefore R = e^{\int P dx}$ which is a required integrating factor (i.e. I.F.)

Now, multiplying both the sides of equation (1) by I.F. We get

$$[e^{\int P dx}] \frac{dy}{dx} + P [e^{\int P dx}] y = Q [e^{\int P dx}]$$

$$\text{or, } \frac{d}{dx}(y \cdot e^{\int P dx}) = Q(e^{\int P dx})$$

On integration, we get

$$y \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} + c$$

which is the required solution.

Working Rules

1. Convert the given differential equation in the form $\frac{dy}{dx} + Py = Q$
2. Find I.F. by integrating P with respect to x i.e. $\int P dx$ and make the power of e, i.e. I.F. = $e^{\int P dx}$.
3. Multiply both sides by I.F. that is $\frac{d}{dx}(y \times \text{I.F.}) = Q \times \text{I.F.}$
4. Integrate to find the required solution as $y \times \text{I.F.} = \int Q \times \text{I.F.} dx + c$.

Note: If the linear differential equation is of the form $\frac{dx}{dy} + Px = Q$, where, P & Q are functions of y (not of x), then I.F. = $e^{\int P dy}$ and solution will be $x \times \text{I.F.} = \int Q \times \text{I.F.} dy + c$.

Example: 1

Solve: $\frac{dy}{dx} + \frac{y}{x} = 2x$

Solution:

Here given differential equation is,

$$\frac{dy}{dx} + \frac{y}{x} = 2x \quad (1)$$

Comparing it with $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{1}{x} \text{ and } Q = 2x$$

Now, the integrating Factor (I.F.) = $e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$.

The general solution is given by

$$y (\text{I.F.}) = \int Q (\text{I.F.}) dx + c$$

$$\text{or, } y(x) = \int 2x \cdot (x) dx + c$$

$$\text{or, } yx = 2 \int x^2 dx$$

$$\text{or, } yx = 2 \frac{x^3}{3} + c$$

$$\therefore y = \frac{2x}{3} + \frac{c}{x}$$

Example: 2

Solve: $x \frac{dy}{dx} + y = x^4$

Solution:

The given differential equation is

$$x \frac{dy}{dx} + y = x^4 \quad (1)$$

which can be expressed as

$$\frac{dy}{dx} + \frac{1}{x} y = x^3$$

Comparing it with $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{1}{x} \text{ and } Q = x^3$$

Here, the integrating Factor (I.F.) = $e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$.

Now, the general solution of equation (1) is given by

$$y (\text{I.F.}) = \int Q (\text{I.F.}) dx + c$$

$$\text{or, } y(x) = \int x^3 \cdot x dx + c$$

$$\text{or, } xy = \int x^4 dx + c$$

$$\text{or, } xy = \frac{x^5}{5} + c$$

$$\therefore y = \frac{x^4}{5} + \frac{c}{x}$$

which is the required solution of equation (1).

Example: 3

Solve: $(1 + x^2) \frac{dy}{dx} + 2xy = 4x^2$

Solution:

To reduce it into standard form of linear equations, we divide it by $(1 + x^2)$ on both the sides.

$$\frac{dy}{dx} + \left(\frac{2x}{1+x^2} \right) y = \frac{4x^2}{1+x^2} \quad (1)$$

Now, by comparing this equation with $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{2x}{1+x^2} \text{ and } Q = \frac{4x^2}{1+x^2}$$

$$\text{or, } \int P dx = \int \frac{2x}{1+x^2} dx$$

$$= \log(1+x^2)$$

$$\therefore \text{I.F.} = e^{\int P dx} = e^{\log(1+x^2)} = 1+x^2$$

Multiplying both sides of equation (1) by I.F., we get

$$(1+x^2) \frac{dy}{dx} + (1+x^2) \frac{2x}{1+x^2} \cdot y = (1+x^2) \frac{4x^2}{(1+x^2)}$$

$$\text{or, } \frac{d}{dx} [(1+x^2)y] = 4x^2$$

On integration, we get

$$y(1+x^2) = \frac{4}{3}x^3 + c.$$

Example: 4

Solve: $y \frac{dx}{dy} = x + 2y^3$

Solution:

The given equation is in the form of $\frac{dx}{dy} + Px = Q$

where P and Q are the functions of y .

$$\text{Here, } \frac{dx}{dy} - \frac{x}{y} = 2y^2 \quad (1)$$

$$\text{Thus, } P = -\frac{1}{y} \text{ and I.F.} = e^{\int P dy} = e^{\int \left(-\frac{1}{y}\right) dy} = e^{-\log y} = \frac{1}{y}$$

Multiplying both sides of (1) by $\frac{1}{y}$, we get

$$\frac{1}{y} \frac{dx}{dy} - \frac{1}{y^2} x = 2y$$

$$\text{or, } \frac{d}{dy} \left(x \cdot \frac{1}{y} \right) = 2y$$

On integration, we get

$$\frac{x}{y} = \int 2y dx$$

$$\text{or, } \frac{x}{y} = y^2 + c$$

$$\text{i.e. } x = y^3 + cy$$

Note:

The above equation can also be written as $(x + 2y^3) \frac{dx}{dy} = y$ and solve accordingly if it shows the easy way.

Example: 5

$$2 \cos x \frac{dy}{dx} + 4y \sin x = \sin 2x$$

Solution: The given differential equation is : $2 \cos x \frac{dy}{dx} + 4y \sin x = \sin 2x$

Dividing both sides by $2 \cos x$ we get,

$$\text{or, } \frac{dy}{dx} + 2y \tan x = \sin x \quad (1)$$

Comparing it with $\frac{dy}{dx} + Py = Q$, we get

$$P = 2 \tan x \text{ and } Q = \sin x$$

Here, the integrating Factor (I.F.) = $e^{\int 2 \tan x dx} = e^{2 \log \sec x} = e^{\log \sec^2 x} = \sec^2 x$.

Multiplying both sides of equation (1) by I.F. and simplifying, we get.

$$\sec^2 x \frac{dy}{dx} + 2y \sec^2 x \tan x = \sec^2 x \sin x$$

$$d(y \sec^2 x) = \sin x \cdot \sec^2 x dx$$

On integration, $\int d(y \sec^2 x) = \int \sec x \cdot \tan x dx$

i.e. $y \sec^2 x = \sec x + c$ which is the required general solution of equation (1).

Example: 6

$$\cos x \frac{dy}{dx} + y \sin x = \sec^3 x$$

Solution:

$$\text{Here, } \frac{dy}{dx} + y \tan x = \sec^3 x$$

Comparing we get

$$P = \tan x \text{ and } Q = \sec^3 x$$

Here, I.F. = $e^{\int P dx} = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$

The general solution is $y (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dx + c$

$$\text{or, } y \sec x = \int \sec^3 x \cdot \sec x dx + c$$

$$\text{or, } y \sec x = \int \sec^4 x dx + c = \int (1 + \tan^2 x) \sec^2 x dx + c$$

$$\text{or, } y \sec x = \int \sec^2 x \, dx + \int \tan^2 x \sec^2 x \, dx + c$$

$$\text{or, } y \sec x = \tan x + \int y^2 \, dy + c \quad \text{for } y = \tan x \text{ (say)}$$

$$\text{or, } y \sec x = \tan x + \frac{y^3}{3} + c$$

$$\therefore y \sec x = \tan x + \frac{1}{3} \tan^3 x + c \text{ which is the required general solution.}$$

Example: 7

$$x(x^2 + 1) \frac{dy}{dx} = y(1 - x^2) + x^2 \log x$$

Solution:

$$\text{The given equation can be written as } \frac{dy}{dx} + \frac{x^2 - 1}{x(x^2 + 1)} y = \frac{x^2 \log x}{x^2 + 1}$$

which is in the form of $\frac{dy}{dx} + Py = Q$ comparing we get

$$P = \frac{x^2 - 1}{x(x^2 + 1)} \text{ and } Q = \frac{x^2 \log x}{x^2 + 1}$$

Here, I.F. = $e^{\int P dx}$

$$\text{Where, } \int P dx = \int \frac{x^2 - 1}{x(x^2 + 1)} dx = \int \frac{(x^2 - 1)}{x^2(x^2 + 1)} x \, dx$$

$$\text{Put, } x^2 = t \quad \therefore 2x dx = dt$$

$$\therefore \int P dx = \frac{1}{2} \int \frac{t - 1}{t(t + 1)} dt = \int \left(\frac{1}{t + 1} - \frac{1}{2t} \right) dt \quad [\because \text{by partial fraction}]$$

$$= \log(t + 1) - \frac{1}{2} \log t = \log \frac{t + 1}{\sqrt{t}} = \log \frac{x^2 + 1}{x}$$

$$\therefore \text{I.F.} = e^{\log \left(\frac{x^2 + 1}{x} \right)} = \frac{x^2 + 1}{x}$$

Multiplying by I.F. and integrating, we get

$$\begin{aligned} y \cdot \frac{x^2 + 1}{x} &= \int \frac{x^2 \log x}{x^2 + 1} \cdot \frac{x^2 + 1}{x} dx \\ &= \int x \log x \, dx = \frac{x^2}{2} \cdot \log x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx \quad [\because \text{Integrating by parts}] \end{aligned}$$

$$\therefore y \left(\frac{x^2 + 1}{x} \right) = \frac{x^2}{2} \log x - \frac{x^2}{4} + c$$

Example: 8

$$\text{Solve: } \frac{dy}{dx} = x^3 y^3 - xy$$

Solution: Here, the given differential equation is $\frac{dy}{dx} = x^3 y^3 - xy$

Dividing by y^3 on both sides we get

$$\text{or, } \frac{1}{y^3} \frac{dy}{dx} = x^3 - \frac{x}{y^2}$$

$$\text{or, } \frac{1}{y^3} \frac{dy}{dx} + \frac{x}{y^2} = x^3 \quad (1)$$

Put $\frac{1}{2y^2} = v$ so that $\frac{1}{y^3} \frac{dy}{dx} = \frac{dv}{dx}$. Thus, equation (1) becomes as,

$$\frac{dv}{dx} + 2xv = x^3 \quad \left[\because \frac{2x}{2y^2} = 2xv \right]$$

Comparing it with $\frac{dv}{dx} + pv = Q$ we get

$$P = 2x \text{ and } Q = x^3$$

$$\therefore \text{I.F.} = e^{\int p dx} = e^{\int 2x dx} = e^{x^2} = e^{x^2}$$

The general solution of the equation is given by

$$v (\text{I.F.}) = \int R(\text{I.F.}) dx + c$$

$$\text{or, } v \cdot e^{x^2} = \int x^3 e^{x^2} dx + c \text{ putting } x^2 = t \text{ then } 2x dx = dt \text{ i.e. } x dx = \frac{1}{2} dt$$

$$\text{or, } v \cdot e^{x^2} = \int x^3 e^{x^2} x dx + c$$

$$\text{or, } v \cdot e^{x^2} = \int t e^t \frac{1}{2} dt + c$$

$$\text{or, } v \cdot e^{x^2} = \frac{1}{2} [te^t - \int (1) e^t dt] + c$$

$$\text{or, } v \cdot e^{x^2} = \frac{1}{2} [te^t - e^t] + c$$

$$\text{or, } v \cdot e^{x^2} = \frac{1}{2} e^{x^2} [x^2 - 1] + c$$

$$\text{or, } v \cdot e^{x^2} = \frac{1}{2} e^{x^2} [x^2 - 1] + \frac{2c}{e^{x^2}}$$

$$\text{or, } \frac{1}{2y^2} = \frac{1}{2} (x^2 - 1 + \frac{2c}{e^{x^2}})$$

$$\therefore \frac{1}{y^2} = x^2 - 1 + 2c e^{-x^2}$$

which is the required general solution of the given differential equation.

Example: 9

Solve: $\frac{dy}{dx} + ay = e^{mx}$, $y(0) = 0$

Solution: Here, $\frac{dy}{dx} + ay = e^{mx}$ (1)

Comparing it with $\frac{dy}{dx} + Py = Q$ we get, $P = a$ and $Q = e^{mx}$

Now,

$$\text{I.F.} = e^{\int p dx} = e^{\int a dx} = e^{ax} = e^{ax}$$

$\therefore$ The general solution is $y (\text{I.F.}) = \int Q (\text{I.F.}) dx + c$

$$\text{or, } y \cdot e^{ax} = \int e^{mx} e^{ax} dx + c = \int e^{(m+a)x} dx + c$$

$$\text{or, } y \cdot e^{ax} = \frac{e^{(m+a)x}}{(m+a)} + c$$

$$\text{or, } y = \frac{e^{mx} \cdot e^{ax}}{e^{ax} (m+a)} + \frac{c}{e^{ax}}$$

$$\therefore y = \frac{e^{mx}}{(a+m)} + ce^{-ax} \quad (2)$$

Given that as $x = 0 \Rightarrow y = 0$

$$\text{So, } 0 = \frac{e^{(0)m}}{(a+m)} + ce^{-a(0)} = \frac{1}{a+m} + c$$

$$\Rightarrow c = -\frac{1}{a+m} \quad [\text{from (2)}]$$

$$y = \frac{e^{mx}}{a+m} - \frac{e^{-ax}}{a+m}$$

EXERCISE - 4(III)

Difficulty Level: A

1. Solve the following linear differential equations:

(a) $\frac{dy}{dx} + \frac{y}{x} = 0$

(b) $dy = (5y - 10)dx$

(c) $\frac{dy}{dx} + \frac{y}{x^2} = \frac{1}{x^2}$

(d) $\frac{dy}{dx} - 2xy = x$

(e) $\frac{dy}{dx} + 2y = 6e^x$

(f) $\tan x \frac{dy}{dx} + y = \sec x$

Difficulty Level: B

2. Solve the following linear differential equations:

(a) $\frac{dy}{dx} + \frac{1}{x}y = 4x^2$

(b) $\frac{dy}{dx} - \frac{y}{x} = 3x^3$

(c) $x \frac{dy}{dx} + 2y = x^2$

(d) $(x^2 - 1) \frac{dy}{dx} + 2xy = 1$

(e) $x \frac{dy}{dx} + y = \log x$

(f) $x \log x \frac{dy}{dx} + y = 2 \log x$

(g) $\sin x \frac{dy}{dx} + y \cos x = x \sin x$

(h) $\cos^2 x \frac{dy}{dx} + y = 1$

(i) $\frac{dy}{dx} + y \tan x = y^3 \sec x$

3. $x \frac{dy}{dx} + y = x^2$ for $y(x) = 0$ when $x = 3$.

ANSWERS

1. (a) $y = \frac{c}{x}$

(b) $y = 2 + c.e^{5x}$

(c) $y = 1 + ce^{1/x}$

(d) $y = -\frac{1}{2} + ce^{x^2}$

(e) $y = 2e^x + ce^{-2x}$

(f) $y \sin x = x + c$

2. (a) $y = x^3 + \frac{c}{x}$

(b) $y = x^4 + cx$

(c) $y = \frac{1}{4}x^2 + \frac{1}{x^2}c$

(d) $y(x^2 - 1) = x + c$

(e) $y = (\log x - 1) + \frac{c}{x}$

(f) $y = \log x + \frac{c}{\log x}$

(g) $y = -x \cot x + 1 + c \operatorname{cosec} x$

(h) $y = 1 + ce^{-\tan x}$

(i) $\cos^2 x = (c - 2 \sin x)y^2$

3. $y = \frac{x^2}{3} - \frac{9}{x}$

4.10 Exact Differential Equation

A first order differential equation $M(x, y) dx + N(x, y) dy = 0$, where M and N are functions of x or y or both, is said to be exact if there is a function $f(x, y)$ such that

$$M(x, y) dx + N(x, y) dy = df(x, y).$$

Here $df(x, y)$ is known as a total or perfect differential.

For example, the differential equation $ydx + xdy = 0$ is exact because the product rule of differentiation gives

$$d(xy) = \frac{d}{dx}(xy) dx = \left(y + \frac{dy}{dx}\right) dx = ydx + x \frac{dy}{dx} \cdot dx = ydx + xdy$$

The solution of given differential equation can be obtained by integrating the differential $d(xy) = 0$ which gives $xy = C$, where C is an arbitrary constant.

But the differential equation $xdy - ydx = 0$ is not exact. However, it can be made exact if we multiply both sides by $\frac{1}{x^2}$ to get $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$. Here the factor $\frac{1}{x^2}$ is called the Integrating Factor (I.F.).

If an equation of the form $Mdx + Ndy = 0$ is not exact, it can be made exact multiplying by some function of x and y . Such a multiplier is called an **Integrating Factor** (I.F.).

Although an equation of the form $Mdx + Ndy = 0$ has integrating factors, there is no general method of finding them. For some differential equations, it may be possible to find an integrating factor even by inspection. The following table will be helpful to determine an I.F. by inspection.

Differential equation	I.F.	Total differential
1. $ydx - xdy$	$\frac{1}{y^2}$	$d\left(\frac{x}{y}\right)$
2. $xdy - ydx$	$\frac{1}{x^2}$	$d\left(\frac{y}{x}\right)$
3. $xdy - ydx$	$\frac{1}{x^2 + y^2}$	$d \tan^{-1}\left(\frac{y}{x}\right)$
4. $xdy - ydx$	$\frac{1}{xy}$	$d \ln\left(\frac{y}{x}\right)$
5. $ydx - xdy$	$\frac{1}{x^2 + y^2}$	$d \tan^{-1}\left(\frac{x}{y}\right)$
6. $xdy + ydx$	$\frac{1}{xy}$	$d \ln(xy)$

Condition of exactness : The differential equation $M(x, y) dx + N(x, y) dy$ is said to be exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, where $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ represent the partial derivative with respect to x and y respectively.

Example: 1

Solve: $xdy + (x + y) dx = 0$

Solution:

We have, $xdy + (x + y) dx = 0$

It can be written as

$$(x dy + y dx) + x dx = 0$$

$$d(xy) + d\left(\frac{1}{2}x^2\right) = 0$$

$$\text{or, } d\left(xy + \frac{1}{2}x^2\right) = 0$$

On integration, we get

$$\therefore xy + \frac{1}{2}x^2 = 0$$

Example: 2

$$\text{Solve: } 2xy \, dy - y^2 \, dx = 0$$

Solution:

We have,

$$2xy \, dy - y^2 \, dx = 0$$

Dividing both sides by x^2 we get

$$\frac{2xy \, dy - y^2 \, dx}{x^2} = 0 \Rightarrow d\left(\frac{y^2}{x}\right) = 0$$

On integrating,

$$\frac{y^2}{x} = C \quad \therefore y^2 = Cx$$

Example: 3

$$(x + y^2) \, dx - 2xy \, dy = 0$$

Solution:

$$\text{We have, } (x + y^2) \, dx - 2xy \, dy = 0$$

$$\text{or, } x \, dx + (y^2 \, dx - 2xy \, dy) = 0$$

Dividing both sides by x^2 , we get

$$\frac{1}{x} \, dx + \frac{y^2 \, dx - 2xy \, dy}{x^2} = 0$$

$$\text{or, } \frac{1}{x} \, dx - \frac{2xy \, dy - y^2 \, dx}{x^2} = 0$$

$$\text{or, } d(\ln x) - d\left(\frac{y^2}{x}\right) = 0$$

On integrating,

$$\ln x - \frac{y^2}{x} = C$$

$$\therefore y^2 = x \ln x - Cx.$$

Example: 4

$$\text{Solve: } \frac{dy}{dx} = \frac{x - y + 1}{x + y + 1}$$

Solution:

$$\text{We have, } \frac{dy}{dx} = \frac{x - y + 1}{x + y + 1}$$

It can be written as

$$(x + y + 1) dy = (x - y + 1) dx$$

$$\text{or, } xdy + ydy + dy = xdx - ydx + dx$$

$$\text{or, } (xdy + ydx) + ydy - xdx + dy - dx = 0$$

$$\text{or, } d(xy) + d\left(\frac{1}{2}y^2\right) - d\left(\frac{1}{2}x^2\right) + dy - dx = 0$$

$$\text{or, } d\left(xy + \frac{1}{2}y^2 - \frac{1}{2}x^2 + y - x\right) = 0$$

On integrating,

$$xy + \frac{1}{2}y^2 - \frac{1}{2}x^2 + y - x = C$$

$$\text{or, } 2xy + y^2 - x^2 + 2y - 2x = 2C$$

$$\therefore 2xy + y^2 - x^2 + 2y - 2x = K$$

where, $K = 2C$ is an arbitrary constant.

Example: 5

Show that the equation is exact and solve it:

$$(x + y - 1) dx + (x - y - 2) dy = 0$$

[CTEVT 2071, 2073]

Solution:

Here, the given equation is,

$$(x + y - 1) dx + (x - y - 2) dy = 0$$

Comparing it with $Mdx + Ndy = 0$, we have

$$M = x + y - 1 \text{ and}$$

$$N = x - y - 2$$

Now, $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 1$. As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so the given equation is exact.

The solution of the differential equation is given by,

$$\int M dx \text{ (taking } y \text{ constant)} + \int N dy \text{ (terms not containing } x) = 0$$

$$\Rightarrow \int (x + y - 1) dx + \int (-y - 2) dy = 0$$

$$\Rightarrow \frac{x^2}{2} + xy - x + C_1 - \frac{y^2}{2} - 2y + C_2 = 0$$

$$\therefore \text{ The required solution is } \frac{x^2}{2} + xy - x - \frac{y^2}{2} - 2y = C \text{ where } C = C_1 + C_2$$

Alternatively Method,

$$(x + y - 1) dx + (x - y - 2) dy = 0$$

$$\text{or, } xdx + ydy - dx + xdy - ydy - 2dy = 0$$

$$\text{or, } \frac{1}{2}d(x^2) - \frac{1}{2}d(y^2) + d(xy) - dx - 2dy = 0$$

$$\text{or, } d\left(\frac{x^2}{2} - \frac{y^2}{2}\right) + d(xy) - d(x + 2y)$$

$$\text{or, } d\left\{\frac{x^2}{2} - \frac{y^2}{2} + xy - (x + 2y)\right\} = 0$$

$$\therefore \frac{x^2}{2} - \frac{y^2}{2} + xy - x - 2y = C$$

EXERCISE - 4(IV)**Difficulty Level: A**

1. Solve the followings on reducing into the exact form:

(a) $xdy - ydx = 0$

(b) $2xy dx - x^2 dy = 0$

(c) $xdy + (x + y) dx = 0$

(d) $(x + y) dy + (y - x) dx = 0$

(e) $2xy dy - y^2 dx + 3x^2 dx = 0$

(f) $\frac{dy}{dx} = \frac{y-x+1}{y-x+5}$

(g) $y(1 + xy) dx - x dy = 0$

Difficulty Level: B

2. Show that the following differential equations are exact and solve them:

(a) $(2ax + by) ydx + (ax + 2by) xdy = 0$

(b) $(xy^2 + x)dx + (yx^2 + y) dy = 0$

(c) $(x^3 + 3xy^2) dx + (3x^2y + y^3) dy = 0$

(d) $(y^2 - x^2) dx + 2xydy = 0$

(e) $e^y dx + (xe^y + 2y) dy = 0$

(f) $(x^2 + xy^2)dx + (x^2y + y^2) dy = 0$

(g) $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$

(h) $xdy - ydx = (x^2 + y^2)dx$

(i) $(2xy + e^y) dx + (x^2 + xe^y) dy = 0$

(j) $(5x^4 + 3x^2y^2 - 2xy^3) dx + (2x^3y - 3x^2y^2 - 5y^4) dy = 0$

ANSWERS

1. (a) $y = Cx$

(b) $x^2 = Cy$

(c) $xy + \frac{x^2}{2} = C$

(d) $2xy + y^2 - x^2 = C$

(e) $y^2 = Cx - 3x^2$

(f) $x^2 + y^2 + 10y - 2x = C$

(g) $\frac{x}{y} + \frac{x^2}{2} = C$

2. (a) $ax^2y + bxy^2 = c$

(b) $x^2y^2 + x^2 + y^2 = c$

(c) $x^4 + 6x^2y^2 + y^4 = c$

(d) $xy^2 - \frac{x^3}{3} = c$

(e) $y^2 = c - xe^y$

(f) $2x^3 + 2y^3 + 3x^2y^2 = c$ where $c = 6c_1$

(g) $\tan x \tan y = c$

(h) $y = x \tan(x + c)$

(i) $x^2y + xe^y = c$

(j) $x^5 + x^3y^2 - x^2y^3 - y^5 = C$

□□□

4.11 Simple Applications of First Order Differential Equation

The applied mathematics is a tool which is used to solve the mathematical problems practically. For that it needs to develop (and to know) the rate of change of quantities with respect to their phenomena (or surroundings). Mathematically we denote them by their respective derivatives and then it need to convert into the mathematical model in terms of differential equation. Here according to the syllabus, it need to do the modeling of first order differential equations only.

We recall that $Mdx + Ndy = 0$ is the first order and first degree differential equation, where M and N are functions of x and y . It can be written as

$$\frac{dy}{dx} = -\frac{M}{N}$$

There are various applications of differential equation in engineering, physics, chemistry, business, economics and so on. Many real world problems in engineering and physical sciences can be modeled in terms of differential equation, for example Newton's law of cooling, population growth and decay, prey and predator problem, harvesting renewable natural resources, electric circuits, orthogonal trajectories, falling body problems etc.

The higher level formulations and applications of differential equations are beyond of this level. Here, we present some of the simple applications of first order and first degree differential equations in the following examples and problems.

4.12 Applications in Coordinate Geometry

Example 1

If the slope of a curve at every point is constant, then show that the curve is a straight line.

Solution:

According to the question, the slope of a curve is constant. Therefore, we have $\frac{dy}{dx} = m$ (constant)

i.e. $dy = m dx$

Integrating, we get $y = mx + c$,

This is a straight line, where c is any constant.

Example 2

If the slope of a curve at a point varies as the ordinate of the point, find the equation of the curve.

Solution:

According to the question, the slope of the curve at a point varies according to the ordinate of the point.

That is, we have $\frac{dy}{dx} = ky$, (k is any constant)

i.e., $\frac{dy}{y} = k dx$ [$\because$ Separating the variables]

Integrating, we get $\log y = kx + \log c$

i.e., $\log y = \log e^{kx} + \log c$

i.e., $\log y = \log (ce^{kx})$

$\therefore y = ce^{kx}$

Example 3

If the normal at every point of a curve passes through a fixed point, show that the curve is a circle.

Solution:

We know that the equation of normal to the curve at any point (x, y) is $Y - y = -\frac{dx}{dy}(X - x)$

If it passes through a fixed point (x_1, y_1) , then $y_1 - y = -\frac{dx}{dy}(x_1 - x)$

$$\text{i.e. } (y_1 - y) dy + (x_1 - x) dx = 0$$

Integrating, we get

$$\frac{1}{2}(y_1 - y)^2 + \frac{1}{2}(x_1 - x)^2 = c$$

$$\text{i.e. } (y - y_1)^2 + (x - x_1)^2 = 2c$$

$$\therefore (x - x_1)^2 + (y - y_1)^2 = r^2, \text{ where } 2c = r^2$$

This is the equation of circle.

Example 4

If the radius of a circle is increasing at a uniform rate of 2 cm/sec, find the radius of the circle after 10 secs if the initial radius is 6 cm.

Solution:

Initially the radius of circle is $OA = r = 6$ cm. Let $OB = R$ be the radius of the circle after 10 second. The change in radius is $\frac{dR}{dt} = 2$ cm/sec

$$\text{i.e., } \frac{dR}{dt} = 2$$

$$\text{i.e., } dR = 2dt$$

Integrating, we get

$$R = 2t + c$$

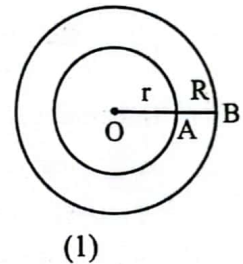
Initially, when $t = 0$, then $R = r = 6$

$$\therefore c = 6$$

Putting $c = 6$ in (1), we get

$$R = 2 \times 10 + 6 = 26$$

Thus the radius after 10 second is 26 cm.

**Example 5**

If the portion of the tangent to a curve included between the coordinate axes, is bisected by the point of contact, show that the curve is rectangular hyperbola.

Solution:

Let the tangent to the curve $y = f(x)$ cuts the x -axis at A and the y -axis at B. Let $P(x, y)$ be the point which bisects line segment AB. The equation of tangent at $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x) \quad (1)$$

Since it passes through $A(X, 0)$,

$$-y = \frac{dy}{dx}(X - x) \quad (2)$$

$$\therefore X = x - \frac{y}{\frac{dy}{dx}}$$

Again line (1) passes through $B(0, y)$

$$Y - y = \frac{dy}{dx}(-x)$$

$$\text{or, } Y = y - x \frac{dy}{dx} \quad (3)$$

Since P is midpoint of AB, we have

$$x = \frac{X + 0}{2} \Rightarrow 2x = x - \frac{y}{\frac{dy}{dx}}$$

$$\text{and } y = \frac{0 + y}{2} \Rightarrow 2y = y - x \frac{dy}{dx}$$

Both of these results give

$$x \frac{dy}{dx} = -y$$

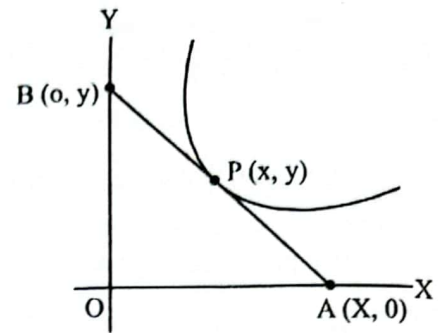
$$\text{i.e., } \frac{dy}{y} + \frac{dx}{x} = 0$$

Integrating, $\log y + \log x = \log c^2$

$$\text{i.e., } \log xy = \log c^2$$

$$\therefore xy = c^2$$

which is the rectangular hyperbola.



4.13 Application in Population Growth

We consider the growth of a population (either of human, animal or bacteria etc) as a function of time t .

Let $x(t)$ be the size of the population at any time t . Though $x(t)$ is integer valued, we approximate $x(t)$ as a differentiable function.

Assume that the population grows at a rate directly proportional to the volume of population present at that time. Therefore, we have

$$\frac{dx}{dt} = kx \quad (1)$$

where k is the constant of proportionality. We consider $k > 0$ since the population is always increasing factor.

The equation (1) can be written as

$$\frac{dx}{x} = k dt$$

Integrating, we get

$$\log x = kt + \log c$$

$$\text{i.e., } \log x = \log e^{kt} + \log c$$

$$\text{i.e., } x = ce^{kt}$$

$$\therefore x(t) = ce^{kt} \quad (2)$$

The value of c and k are determined with the help of initial conditions.

The equation (2) shows that the population increases exponentially with time t . This law of population growth is called Malthusian Law.

Example 6

In the initial stage, there were 5 lakhs bacteria in any decayed substance. Three hours later the number has increased to 9 lakhs. Find the number of bacteria at the end of 12 hours, if their increment rate remains same.

Solution:

Let us tabulate the given problem which makes us easy to solve the problem.

No. of bacteria (in lakhs)	5	9	3
Time in hours	0	3	12

Let y_0 and y represent the initial population and the population after time t hours. According to the given question the growth rate of bacteria is proportional to its size. i.e. $\frac{dy}{dt} \propto y \Rightarrow \frac{dy}{dt} = cy$ (c is proportionality constant)

$$\text{or, } \frac{dy}{y} = c dt.$$

On integrating, we get, $\log y = ct + \log k$ where $\log k$ is an integrating constant.

$$\log y - \log k = ct$$

$$\Rightarrow \frac{y}{k} = e^{ct}$$

$$\therefore y = ke^{ct} \quad (1)$$

Case - I : When $t = 0$ and $y = 5$ then from equation (1) we have,

$$5 = k e^0$$

$$\therefore k = 5$$

$$\text{from (1), we get } y = 5e^{ct} \quad (2)$$

Case - II : When $t = 3$ and $y = 9$

from (2) then we have

$$9 = 5e^{3c}$$

$$\text{or, } e^{3c} = \frac{9}{5} = 1.8$$

$$\text{or, } 3c = \log (1.8)$$

$$\therefore e = 0.1986$$

Again from (2) we get $y = 5e^{0.196t}$

(3)

Case - III when $t = 12$ hrs.

From (3) we get

$$y = 5e^{0.196t} = 5e^{0.196 \times 12} = 5 \times 10.5 = 52.5$$

Thus, the required number of bacteria at the end of 12 hours is 52.5 lakhs.

Example 7

The growth of population is proportional to the number of present population. If the population of a colony doubles in 50 years, in how many years will the population become triple?

Solution:

Let $x(t)$ be the population in time t . Clearly we have $\frac{dx}{dt} = kx$

$$\text{i.e., } \frac{dx}{x} = kdt$$

Integrating, we get

$$\log x = kt + \log c$$

$$\text{i.e., } \log x = \log e^{kt} + \log c$$

$$\text{i.e., } x = ce^{kt} \quad (1)$$

Let x_0 be the population when $t = 0$.

$$\text{Then } x_0 = ce^0 \Rightarrow c = x_0$$

$$\text{From (1), we get } x = x_0 e^{kt}$$

Now $x = 2x_0$ when $t = 50$

$$\text{From (1), } 2x_0 = x_0 e^{50k}$$

$$\text{i.e., } 2 = e^{50k}$$

$$\text{i.e., } 50k = \log_e(2)$$

$$\therefore k = \frac{1}{50} \log 2$$

$$\text{Now from (1), } x = x_0 e^{\frac{t}{50} \log 2}$$

$$\text{i.e., } x = x_0 e^{\log(2) \frac{t}{50}}$$

$$\therefore x = x_0 2^{\frac{t}{50}}$$

This is the population at any time t .

Again assume that the population is tripled in t_1 years.

That is, $x = 3x_0$ when $t = t_1$

$$\text{From (1), } 3x_0 = x_0 e^{\frac{t_1}{50} \log 2}$$

$$\text{i.e., } 3 = e^{\frac{t_1}{50} \log 2}$$

$$\text{i.e., } \log 3 = \frac{t_1}{50} \log 2$$

$$\text{i.e., } t_1 = 50 \frac{\log 3}{\log 2}$$

Therefore, the population is tripled in

$$50 \left(\frac{\log 3}{\log 2} \right) \text{ years.}$$

We recommend the students to explore the other similar applications of differential equation, for example Radioactive decay, Newton's law of cooling/warming and mixture problems.

EXERCISE - 4(V)

Difficulty Level: A

1. If the slope of a curve at a point varies as the abscissa of the point, show that the curve is parabola.
2. If the slope of a curve at a point varies as the reciprocal of twice the ordinate of the point, show that the curve is a parabola. And hence find the equation of the particular parabola which passes through the point (4, 3).
3. If a particle moves in a straight line with uniform acceleration f , starting with an initial velocity u , prove that

$$(a) \quad v = u + ft$$

$$(b) \quad s = ut + \frac{1}{2}ft^2$$

$$(c) \quad v^2 = u^2 + 2fs.$$

where, s and v have usual meanings.

Difficulty Level: B

4. Show that the equation of the curve whose slope at any point is equal to $(y + 2x)$ and passes through the origin is $y^2 = x^2 - x - 1$.
5. If the rate of population growth is 5% per year then how long time does it take to be double?
6. 40 % of wet clothes dry in 1:40 hours. How long time does it take to get dry of 90% clothes?

Answers

2. $y^2 = kx + c ; y^2 - 9 = k(x - 4)$
5. $\log (2)$ years
6. $T = 7 : 30$ hrs



4.14 Partial Differential Equation

An equation involving partial derivatives of a function of two or more variables is known as a partial differential equation (P.D.E.). Partial derivatives, occur only when there are two or more than two independent variables. So, in partial differential equations there are more than one independent variables.

The order of a partial differential equation is the order of the partial differential coefficient of the highest order term appearing in it. The degree of such equation is the greatest exponent of the highest order term. The following are the some examples of order and degree of PDE.

- (a) $x \frac{\partial u}{\partial y} + y \frac{\partial u}{\partial y} = 2u$ [First order and first degree]
- (b) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = c^2 \frac{\partial^2 u}{\partial t^2}$ [Second order and first degree]
- (c) $\left(\frac{\partial^3 u}{\partial x^3}\right)^2 + \left(\frac{\partial^3 u}{\partial x^2}\right)^3 = 0$ [Third order and second degree]

When we consider the case of two independent variables we usually assume them to be x and y and assume z to be the dependent variable. We adopt the following notations throughout the study of partial differential equations.

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$$

Sometimes the partial differentials are also denoted by making use of suffixes. Thus, we write

$$u_x = \frac{\partial u}{\partial x}, u_y = \frac{\partial u}{\partial y}, u_{xx} = \frac{\partial^2 u}{\partial x^2}, u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \text{ and } u_{yx} = \frac{\partial^2 u}{\partial y \partial x}$$

4.15 Formation of Partial Differential Equations

We shall now examine the interesting equation that how partial differential equations arise. We show that such equations can be formed by elimination of arbitrary constants or that of arbitrary functions.

(a) By the elimination of arbitrary constant

Consider an equation $f(x, y, z, a, b) = 0$ (1)

where a and b are arbitrary constants. Let z be regarded as function of two independent variables x and y .

Differentiating (1) partially with respect to x and y , we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

$$\text{or, } \frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} = 0 \quad (2)$$

$$\text{and } \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$$

$$\text{or, } \frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z} = 0 \quad (3)$$

Eliminating a and b from (1), (2) and (3), we have differential equation of the form

$$F(x, y, z, p, q) = 0$$

which is the partial differential equation of first order.

Example: 1

Form a partial differential equation by eliminating a, b from $Z = (x + a)(y + b)$

Solution:

$$\text{We have, } Z = (x + a)(y + b) \quad (1)$$

Differentiating (1) partially with respect to x and y , we get

$$\frac{\partial z}{\partial x} = y + b$$

$$\text{and } \frac{\partial z}{\partial y} = x + a$$

Substituting in (1), we have

$$Z = \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}$$

$$\Rightarrow Z = pq$$

which is the required differential equation.

In this example, we have two independent variables x, y and two constants a, b and the elimination of the constants resulted in a differential equation of first order. If the number of arbitrary constants is more than the number of independent variables then the partial differential equation thus obtained will be of higher order than that of the first as shown below.

Example: 2

Form partial differential equation by eliminating the constant a, b and c from

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

[CTEVT 2067 / 2068]

Solution:

$$\text{We have, } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (1)$$

Differentiating (1) partially with respect to x and y , we get

$$\frac{2x}{a^2} + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial x} = 0$$

$$\text{or, } c^2 x + a^2 z \frac{\partial z}{\partial x} = 0 \quad (2)$$

$$\text{and } \frac{2y}{b^2} + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial y} = 0$$

$$\text{or, } c^2 y + b^2 z \frac{\partial z}{\partial y} = 0 \quad (3)$$

Now, differentiating (2) partially with respect to x , we get

$$c^2 + a^2 \left(\frac{\partial z}{\partial x} \right)^2 + a^2 z \frac{\partial^2 z}{\partial x^2} = 0$$

$$\text{or, } \frac{c^2}{a^2} + \left(\frac{\partial z}{\partial x} \right)^2 + z \frac{\partial^2 z}{\partial x^2} = 0 \quad (4)$$

Substituting $\frac{c^2}{a^2} = -\frac{z}{x} \frac{\partial z}{\partial x}$ from (2) in (4), we get

$$-\frac{z}{x} \frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z \frac{\partial^2 z}{\partial x^2} = 0$$

$$\therefore -z \frac{\partial z}{\partial x} + x \left(\frac{\partial z}{\partial x}\right)^2 + zx \frac{\partial^2 z}{\partial x^2} = 0 \quad (5)$$

Again, differentiating (3) partially with respect to y , we get

$$c^2 + b^2 \left(\frac{\partial z}{\partial y}\right)^2 + b^2 z \frac{\partial^2 z}{\partial y^2} = 0$$

or, $\frac{c^2}{b^2} + \left(\frac{\partial z}{\partial y}\right)^2 + z \frac{\partial^2 z}{\partial y^2} = 0 \quad (6)$

Substituting $\frac{c^2}{b^2} = -\frac{z}{y} \frac{\partial z}{\partial y}$ from (3) in (6), we get

$$-\frac{z}{y} \frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial y}\right)^2 + z \frac{\partial^2 z}{\partial y^2} = 0$$

$$\therefore -z \frac{\partial z}{\partial y} + y \left(\frac{\partial z}{\partial y}\right)^2 + zy \frac{\partial^2 z}{\partial y^2} = 0 \quad (7)$$

The equations (5) and (7) are the partial differential equations of second order and first degree.

Example: 3

Form a partial differential equation if $Z = ke^{ax} \sin ay$.

[CTEVT 2071]

Solution:

Here, $Z = ke^{ax} \sin ay \quad (1)$

partially differentiating (1) with respect to x ,

$$\frac{\partial z}{\partial x} = ke^{ax} \sin ay \quad (2)$$

Again, partially differentiating (1) with respect to y , we have

$$\frac{\partial z}{\partial y} = kae^{ax} \cos ay \quad (3)$$

Again, partially differentiating (2) with respect to x and (3) with respect to y respectively we have,

$$\frac{\partial^2 z}{\partial x^2} = ka^2 e^{ax} \sin ay \quad (4)$$

and $\frac{\partial^2 z}{\partial y^2} = -ka^2 e^{ax} \sin ay \quad (5)$

Adding (4) and (5) we get,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = ka^2 e^{ax} \sin ay - ka^2 e^{ax} \sin ay = 0$$

or, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ which is the required equation.

(b) By the elimination of arbitrary function ϕ from the equation $f(u, v) = 0$

We have the equation

$$f(u, v) = 0 \quad (1)$$

where u and v are functions of x, y and z . We treat z as dependent variable and x and y as independent variables, so that

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial v}{\partial x} = 0, \frac{\partial x}{\partial y} = 0,$$

Differentiating (1) partially with respect to x , we get

$$\frac{\partial \phi}{\partial u} \left(\frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x} \right) + \frac{\partial \phi}{\partial v} \left(\frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial v}{\partial z} \cdot \frac{\partial z}{\partial x} \right) = 0$$

$$\text{or, } \frac{\partial \phi}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial z}{\partial x} \cdot \frac{\partial u}{\partial z} \right) + \frac{\partial \phi}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial z}{\partial x} \cdot \frac{\partial v}{\partial z} \right) = 0$$

$$\text{or, } \frac{\partial \phi}{\partial u} \left(\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} \right) + \frac{\partial \phi}{\partial v} \left(\frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z} \right) = 0$$

$$\text{or, } \frac{\partial \phi / \partial u}{\partial \phi / \partial v} = \frac{\left(\frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z} \right)}{\left(\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} \right)} \quad (2)$$

Similarly differentiating (1) partially with respect to y , we get

$$\frac{\partial \phi / \partial u}{\partial \phi / \partial v} = \frac{\left(\frac{\partial v}{\partial y} + q \frac{\partial v}{\partial z} \right)}{\left(\frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} \right)} \quad (3)$$

Eliminating ϕ with the help of (2) and (3), we get

$$\frac{\left(\frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z} \right)}{\left(\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} \right)} = \frac{\left(\frac{\partial v}{\partial y} + q \frac{\partial v}{\partial z} \right)}{\left(\frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} \right)}$$

$$\text{or, } \left(\frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z} \right) \left(\frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} \right) = \left(\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} \right) \left(\frac{\partial v}{\partial y} + q \frac{\partial v}{\partial z} \right)$$

Which can be written as

$$\left(\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial v}{\partial y} \cdot \frac{\partial u}{\partial z} \right) p + \left(\frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial z} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z} \right) q = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}$$

$$\text{or, } Pp + Qq = R$$

Where,

$$P = \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial v}{\partial y} \cdot \frac{\partial u}{\partial z}$$

$$Q = \frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial z} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z}$$

$$R = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}$$

Which is the required partial differential equation.

Example: 4

Obtain a partial differential equation by eliminating the function f from

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right).$$

Solution:

$$\text{We have, } z = y^2 + 2f\left(\frac{1}{x} + \log y\right). \quad (1)$$

Differentiating (1) partially with respect to x , we get

$$\begin{aligned} \frac{\partial z}{\partial x} &= 2f'\left(\frac{1}{x} + \log y\right) \cdot \frac{\partial}{\partial x}\left(\frac{1}{x} + \log y\right) \\ \text{or, } p &= 2f'\left(\frac{1}{x} + \log y\right) \cdot \left(-\frac{1}{x^2}\right) \\ \text{or, } -px^2 &= 2f'\left(\frac{1}{x} + \log y\right) \end{aligned} \quad (2)$$

Again, differentiating (1) partially with respect to y , we get

$$\begin{aligned} \frac{\partial z}{\partial y} &= 2y + 2f'\left(\frac{1}{x} + \log y\right) \cdot \frac{\partial}{\partial y}\left(\frac{1}{x} + \log y\right) \\ \text{or, } q &= 2y + 2f'\left(\frac{1}{x} + \log y\right) \cdot \frac{1}{y} \\ \text{or, } qy - 2y^2 &= 2f'\left(\frac{1}{x} + \log y\right) \end{aligned} \quad (3)$$

From (2) and (3), we have

$$\begin{aligned} -px^2 &= qy - 2y^2 \\ \text{or, } px^2 + qy &= 2y^2 \end{aligned}$$

which is the required partial differential equation.

Example: 5

Form a partial differential equation by eliminating the arbitrary function f and ϕ from

$$z = f(x + ay) + \phi(x - ay).$$

Solution:

$$\text{We have, } z = f(x + ay) + \phi(x - ay). \quad (1)$$

Differentiating (1) twice partially with respect to x , we get

$$\begin{aligned} \frac{\partial z}{\partial x} &= f'(x + ay) \frac{\partial}{\partial x}(x + ay) + \phi'(x - ay) \frac{\partial}{\partial x}(x - ay) \\ &= f'(x + ay) + \phi'(x - ay) \\ \text{Again, } \frac{\partial^2 z}{\partial x^2} &= f''(x + ay) \frac{\partial}{\partial x}(x + ay) + \phi''(x - ay) \frac{\partial}{\partial x}(x - ay) \\ &= f''(x + ay) + \phi''(x - ay) \end{aligned} \quad (2)$$

Again, differentiating (1) twice partially with respect to y , we get

$$\begin{aligned}
\frac{\partial z}{\partial y} &= f'(x+ay) \frac{\partial}{\partial y}(x+ay) + \phi'(x-ay) \frac{\partial}{\partial y}(x-ay) \\
&= a f'(x+ay) + (-a) \phi'(x-ay) \\
\therefore \frac{\partial^2 z}{\partial y^2} &= a f''(x+ay) \frac{\partial}{\partial y}(x+ay) - a \phi''(x-ay) \frac{\partial}{\partial y}(x-ay) \\
&= a^2 f''(x+ay) + a^2 \phi''(x-ay) \\
\frac{1}{a^2} \frac{\partial^2 z}{\partial y^2} &= f''(x+ay) + \phi''(x-ay) \quad (3)
\end{aligned}$$

From (2) and (3),

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{1}{a^2} \frac{\partial^2 z}{\partial y^2} \\
\therefore \frac{\partial^2 z}{\partial x^2} &= \frac{1}{a^2} \frac{\partial^2 z}{\partial y^2} \text{ is the required partial differential equation.}
\end{aligned}$$

Example: 6

Form a P.D.E. eliminating f from the equation $lx + my + nz = f(x^2 + y^2 + z^2)$ [CTEVTE]

Solution:

Here, the given equation is

$$lx + my + nz = f(x^2 + y^2 + z^2) \quad (1)$$

Partially differentiating (1) with respect to x ,

$$l + n \cdot \frac{\partial z}{\partial x} = f'(x^2 + y^2 + z^2) \cdot (2x + 2z \frac{\partial z}{\partial x})$$

$$\text{or, } l + np = f'(x^2 + y^2 + z^2) \cdot (2x + 2zp) \quad (2)$$

Again, partially differentiating (1) with respect to y ,

$$m + n \cdot \frac{\partial z}{\partial y} = f'(x^2 + y^2 + z^2) \cdot (2y + 2z \frac{\partial z}{\partial y})$$

$$\text{or, } m + nq = f'(x^2 + y^2 + z^2) \cdot (2y + 2zq) \quad (3)$$

Dividing (2) by (3),

$$\frac{l + np}{m + nq} = \frac{2x + 2zp}{2y + 2zq} = \frac{x + zp}{y + zq}$$

$$\text{or, } (l + np)(y + zq) = (m + nq)(x + zp)$$

$$\text{or, } ly + lqz + npy + npqz = mx + mpz + nqx + npqz$$

$$\text{or, } (ny - mz)p + (lz - nx)q = mx - ly, \text{ which is the required P.D.E.}$$

EXERCISE - 4(VI)**Difficulty Level: A**

1. Obtain partial differential equations by eliminating the arbitrary constant from the followings:

(a) $z = ax + a^2y^2 + b$

(b) $z = ax + by + ab$

(c) $z = ax + by + a^2 + b^2$

(d) $z = (x - a)^2 + (y - b)^2$

(e) $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Difficulty Level: B

2. Eliminate the arbitrary functions and hence obtain the partial differential equations from the followings:

(a) $z = f\left(\frac{y}{x}\right)$

(b) $z = f(x^2 - y^2)$

(c) $z = x + y + f(xy)$

(d) $z = f(x + iy) + f(x - iy)$

(e) $z = xf_1(x + y) + f_2(x + y)$

(f) $x + y + z = f(x^2 + y^2 + z^2)$

ANSWERS

1. (a) $q = 2yp^2$ (b) $z = px + qy + pq$ (c) $z = px + qy + p^2 + q^2$
 (d) $4z = p^2 + q^2$ (e) $2z = px + qy$
2. (a) $px + qy = 0$ (b) $qx + py = 0$ (c) $px - qy = x - y$
 (d) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ (e) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2}$ (f) $p(y - z) + q(z - x) = x - y$

□□□

4.16. Linear PDE of First Order

A differential equation involving partial derivatives (i.e. partial differential coefficients) $\frac{\partial z}{\partial x} = p$ and $\frac{\partial z}{\partial y} = q$ only of first order and not involving higher order is called first order PDE. If the degree of p and q are unity, then it is known as a linear partial differential equation of first order. Otherwise it is known as non-linear partial differential equation of order one.

4.17 Lagrange's Linear PDE

The PDE of the form $Pp + Qq = R$ where P, Q, R are functions of x, y, z is the standard form of the linear partial differential equation of order first and is called Lagrange's linear equation.

From Art 2.11 (b) we have the content that by eliminating an arbitrary function ϕ from

$$f(u, v) = 0 \quad (1)$$

connecting two functions u and v , we get the partial differential equation

$$Pp + Qq = R \quad (2)$$

$$\text{Where, } P = \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial v}{\partial y} \cdot \frac{\partial u}{\partial z}$$

$$Q = \frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial z} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z}$$

$$R = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}$$

$$\text{Where, } p = \frac{\partial z}{\partial x} \text{ and } q = \frac{\partial z}{\partial y}$$

Thus (1) is the general integral of (2) and so we have to find the values of u and v .

For that let $u = f(x, y, z) = a$ and $v = f(x, y, z) = b$ be two equations where a and b are arbitrary constants.

Differentiating them, we get

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0$$

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0$$

Solving these, we get

$$\frac{dx}{\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial v}{\partial y} \cdot \frac{\partial u}{\partial z}} = \frac{dy}{\frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}}$$

$$\text{i.e. } \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad (3)$$

This equation is called Lagrange's auxiliary equations or subsidiary equation.

Solution of the above differential equations are $u = f(x, y, z) = a$ and $v = f(x, y, z) = b$. Thus the solution of the given equation (1) is found as $f(u, v) = 0$.

Working Method

For solving partial differential equation by Lagrange's method, we have following steps :

Step I : Put the given linear PDE of the first order in the standard form

$$Pp + Qq = R \quad (1)$$

Step II : Write down Lagrange's auxiliary equations for (1) namely,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad (2)$$

Step III : Find any two independent solutions of the subsidiary equations. Let them be $u = a$ and $v = b$, where u, v are functions of x, y, z and a, b are constants.

Step IV : Now the general solution (integral) of (1) is given by $f(u, v) = 0$ or $u = f(v)$
or, $v = f(u)$

Example: 1

Solve : $p + q = \frac{z}{a}$.

Solution:

We have, $p + q = \frac{z}{a}$

Comparing with $Pp + Qq = R$, we get

$$P = 1, Q = 1 \text{ and } R = \frac{z}{a}$$

Now, the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{z/a}$$

Taking first two terms,

$$\frac{dx}{1} = \frac{dy}{1}$$

$$\therefore x - y = c_1$$

Again, taking the last two terms

$$dy = \frac{dz}{z/a}$$

$$\text{or, } \frac{dy}{a} = \frac{dz}{z}$$

$$\text{or, } \log z = \frac{1}{a}y + \log c_2$$

$$\text{or, } \frac{y}{a} = \log z - \log c_2$$

$$\text{or, } \frac{y}{a} = \log \frac{z}{c_2}$$

$$\text{or, } \log \frac{z}{c_2} = \frac{y}{a}$$

$$\text{or, } \log \frac{z}{c_2} = \frac{1}{a}y$$

$$\therefore z = c_2 e^{y/a}$$

The general solution is $z = e^{y/a} (x - y)$.

Example: 2

Solve : $yz \frac{\partial z}{\partial x} + zx \frac{\partial z}{\partial y} = xy.$

Solution:

We have, $yz \frac{\partial z}{\partial x} + zx \frac{\partial z}{\partial y} = xy.$

It can be written as

$$yzp + zxq = xy \quad (1)$$

Comparing it with $Pp + Qq = R$, we get

$$P = yz, Q = zx, R = xy$$

Now the Lagrange's auxiliary equations for (1) are

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

Taking first two terms,

$$\frac{dx}{yz} = \frac{dy}{zx} \Rightarrow \frac{dx}{y} = \frac{dy}{x}$$

or, $x dx = y dy$

$$\therefore x^2 - y^2 = c_1$$

Again, taking last two terms

$$\frac{dy}{zx} = \frac{dz}{xy} \Rightarrow \frac{dy}{z} = \frac{dz}{y}$$

or, $y dy = z dz$

$$\therefore y^2 - z^2 = c_2$$

Here, $x^2 - y^2 = c_1$ and $y^2 - z^2 = c_2$ are two independent solutions.

Hence the required general solution is $f(x^2 - y^2, y^2 - z^2) = 0$ where f is an arbitrary function.

Example: 3

Solve : $y^2 z \frac{\partial z}{\partial x} + x^2 z \frac{\partial z}{\partial y} = xy^2.$

Solution:

We have, $y^2 zp + x^2 zq = xy^2.$

Comparing with $Pp + Qq = R$, we get

$$P = zy^2, Q = zx^2, R = xy^2$$

The subsidiary equations are

$$\frac{dx}{zy^2} = \frac{dy}{zx^2} = \frac{dz}{xy^2}$$

Taking first two ratios we have,

$$\frac{dx}{zy^2} = \frac{dy}{zx^2}$$

or, $x^2 dx = y^2 dy$

On integration, it gives

$$x^3 - y^3 = c_1$$

Taking first and last ratios,

$$\frac{dx}{zy^2} = \frac{dy}{xy^2}$$

$$\text{or, } \frac{dx}{z} = \frac{dz}{x}$$

$$\text{or, } xdx = zdz$$

$$\text{or, } x^2 - z^2 = c_2$$

Here, $x^3 - y^3 = c_1$, and $x^2 - z^2 = c_2$ are two independent solutions.

Hence the required general solution $f(x^3 - y^3, x^2 - z^2) = 0$.

Example: 4

$$\text{Solve : } pz - qz = z^2 + (x + y)^2$$

Solution:

$$\text{We have, } pz - qz = z^2 + (x + y)^2$$

Comparing with $Pp + Qq = R$, we get

$$P = z, Q = -z, R = z^2 + (x + y)^2$$

The subsidiary equations are

$$\frac{dx}{z} = \frac{dz}{-z} = \frac{dz}{z^2 + (x + y)^2}$$

Taking first two ratios, we get

$$\frac{dx}{z} = \frac{dz}{-z}$$

$$\text{or, } dx = -dz$$

$$\text{or, } dx + dz = 0$$

$$\therefore x + z = c_1$$

Taking first and last ratios,

$$\frac{dx}{z} = \frac{dz}{z^2 + (x + y)^2}$$

$$\text{or, } \frac{zdz}{z^2 + (x + y)^2} = dx$$

$$\text{or, } \frac{2zdz}{z^2 + c_1^2} = 2dx$$

Integrating,

$$\log(z^2 + c_1^2) = 2x + c_2$$

$$\text{or, } \log[z^2 + (x + y)^2] = 2x + c_2$$

$$\text{or, } \log(z^2 + x^2 + y^2 + 2xy) - 2x = c_1$$

The general solution is

$$f[x + y, \log(z^2 + x^2 + y^2 + 2xy) - 2x] = 0.$$

4.18 Method of Lagrange's Multipliers

If the auxiliary equation of a differential equation is in the form as:

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ then it will be convincing to find the solution of the equation on reducing it as,

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l dx + m dy + n dz}{lP + mQ + nR}$ where l, m, n are known as the Lagrange's multipliers (constants) which are so chosen that $lP + mQ + nR = 0$.

Example: 5

Solve the partial differential equation :

$$xp - yq + x^2 - y^2 = 0.$$

Solution:

The given equation is

$$xp - yq + x^2 - y^2 = 0.$$

It can be written as

$$xp - yq = y^2 - x^2$$

Comparing it with $Pp + Qq = R$ we have,

$$P = x, Q = -y \text{ and } R = y^2 - x^2$$

Now, the auxiliary equation becomes as

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{or, } \frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{y^2 - x^2} \quad (1)$$

Taking the Lagrange's multipliers as x, y, l we get

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{y^2 - x^2} = \frac{xdx + ydy + dz}{0}$$

$$\text{i.e. each ratio} = \frac{xdx + ydy + dz}{0}$$

$$\text{or, } xdx + ydy + dz = 0$$

Integrating we have,

$$\frac{x^2}{2} + \frac{y^2}{2} + z = \frac{C_1}{2}$$

$$\text{or, } x^2 + y^2 + 2z = C_1 \quad (2)$$

Taking the first two ratios from (1)

$$\frac{dx}{x} = \frac{dy}{-y}$$

Integrating we get,

$$\log x = -\log y + \log C_2$$

$$\text{or, } \log x + \log y = \log C_2$$

$$\text{or, } \log(xy) = \log C_2$$

$$\text{or, } (xy) = C_2$$

$\therefore$ The solution of the given equation is

$$(xy) = f(x^2 + y^2 + 2z).$$

Exercise - 4 (VII)

Difficulty Level: A

1. Solve : $p + q = 1$
3. Solve : $x^2p + y^2q = z^2$
5. Solve: $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \sin x$

2. Solve : $xp + yq = z$
4. Solve: $px - qy = 2x$
6. Solve: $xz \frac{\partial z}{\partial x} + yz \frac{\partial z}{\partial y} = xy$

Difficulty Level: B

7. Solve: $(y - z) \frac{\partial z}{\partial x} + (x - y) \frac{\partial z}{\partial y} = z - x$

8. Solve: $(y + z)p + (z + x)q = x + y$

9. Solve: $y^2p - xyq = x(z - 2y)$

10. Solve: $\frac{y^2z}{x}p + zxq = y^2$

11. Solve: $zx \frac{\partial z}{\partial x} - yz \frac{\partial z}{\partial y} = y^2 - x^2$

12. Solve: $(mz - ny)p + (nx - lz)q = ly - mx$

13. Solve: $z(x + y)p + z(x - y)q = x^2 + y^2$

14. Solve: $\left(\frac{y-z}{yz}\right)p + \left(\frac{z-x}{zx}\right)q = \frac{x-y}{xy}$

ANSWERS

1. $\phi(x - y, x - z) = 0$

2. $\phi\left(\frac{x}{z}, \frac{y}{z}\right) = 0$

2. $\phi\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z}\right) = 0$

4. $xy = f(-2x + z)$

5. $z + \cos x = \phi(y - x)$

6. $f\left(\frac{x}{y}, xy - z^2\right) = 0$

7. $f(x + y + z, x^2 - y^2 - 2xy + 2yz) = 0$ or, $f(x + y + z, xy + zx - yz) = 0$

8. $f[x^2 - y^2 + 2zx - 2yz, y^2 - z^2 + 2xy - 2zx] = 0$

9. $x^2 + y^2 = f(y^2 - yz)$

10. $f(x^3 - y^3, x^2 - z^2) = 0$

11. $f(x^2 + y^2 + z^2, xy) = 0$

12. $x^2 + y^2 + z^2 = f(lx + my + nz)$

13. $f(x^2 - y^2 - 2xy, x^2 - y^2 - z^2) = 0$

14. $(xyz, x + y + z) = 0$

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Fourier Series

Unit

5

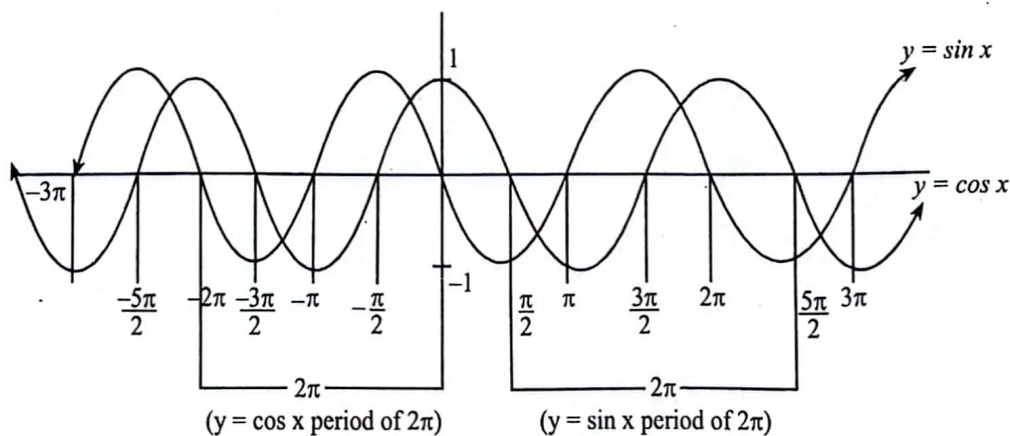
5.1. Introduction

The Fourier series is one of the most powerful tools in the modern mathematics for solving some ordinary and partial differential equations. The concept of the Fourier series was first developed (introduced) by the French physicist and mathematician named Jean-Baptiste Joseph Fourier (1768-1830). He developed the Fourier series and an expansion of a function into an infinite trigonometric series in terms of sine and cosine functions in the process of solving the problems of heat conduction (transfer). As the Taylor series express, a function in terms of algebraic terms, likewise the Fourier series express the functions in terms of the trigonometric terms as: $1, \cos x, \cos 2x, \dots, \sin x, \sin 2x, \sin 3x, \dots$

The theory of Fourier series is a little bit complicated but its application is simple. Fourier series are more universal than that of Taylor's series because many of the discontinuous periodic functions of practical interest can be developed (expressed) in Fourier series. But it does not have the Taylor's series representations. The Fourier series are used to solved the engineering problems of periodic functions related to heat conduction, electromotive forces in electric circuits, acoustics vibration, planetary motion, signal processing, optic etc.

5.2. Periodic Functions

Since $\cos(2\pi + x) = \cos x$, so $\cos x$ is periodic in an interval of length 2π . Now 2π is called a period of $\cos x$ (a trigonometric function). Similarly $\sin x$ and $\tan x$ are also the periodic functions in some interval of length 2π and π respectively. Look at the graph of sine and cosine below which have a periodic repetition in the length of 2π in the interval $(-\infty, \infty)$.



Hence, a function $f(x)$ is said to be periodic with period P if $f(x + P) = f(x)$ for all $(\forall)$ values of x , $P > 0$. For examples,

$$\sin(x + 2\pi) = \sin x, \tan(x + \pi) = \tan x \text{ etc.}$$

5.3. Fundamental Period of a Periodic Function

In a periodic function $f(x)$, the smallest period $P > 0$, is called the fundamental period of $f(x)$. Thus, the fundamental period of $\sin x, \cos x$ is 2π and π is the fundamental period of $\sin 2x, \cos 2x$ and $\tan x$ etc.

Example: 1

Find the fundamental period of $\sin 2x$.

Solution:

Here, $f(x) = \sin 2x$

If P be the fundamental period of $f(x)$ then we have by the definition that $f(x + P) = f(x)$

$$\Rightarrow \sin 2(x + P) = \sin 2x = \sin (2\pi + 2x)$$

$$\text{or, } 2(x + P) = 2\pi + 2x$$

$$\text{or, } 2x + 2P = 2\pi + 2x$$

$$\text{or, } 2P = 2\pi$$

$$\text{or, } P = \pi$$

$\therefore$ The fundamental period of $\sin 2x$ is $P = \pi$.

Notes:

1. If $f(x)$ is a periodic function with period P , then

$$f(x + 2P) = f(x + P + P) = f(x + P) = f(x)$$

$$\therefore f(x + 2P) = f(x)$$

$$\text{and } f(x + 3P) = f[(x + 2P) + P] = f(x + 2P) = f(x)$$

$$\therefore f(x + 3P) = f(x)$$

.....

.....

$$\begin{aligned} f(x + nP) &= f[x + (n - 1)P + P] \\ &= f[(x + (n - 1)P)] = f(x) \end{aligned}$$

$$\text{or, } f(x + nP) = f(x)$$

$$\therefore f(x + nP) = f(x)$$

Hence, $2P, 3P, \dots, nP$ are also the periods of $f(x)$.

2. If $f(x)$ and $g(x)$ are the periodic functions with period P and $h(x) = af(x) + bg(x)$ where a and b are constants then

$$h(x + P) = af(x + P) + bg(x + P) = af(x) + bg(x) = h(x)$$

$$\text{or, } h(x + P) = h(x)$$

$\therefore h(x)$ is also a periodic function with period P .

Some properties of periodic functions

If $f(x)$ is a periodic function of period 2π , i.e. $f(x + 2\pi) = f(x)$, then

$$(a) \quad \int_a^b f(x) dx = \int_{a+2\pi}^{b+2\pi} f(x) dx$$

$$(b) \int_{-\pi}^{\pi} f(x) dx = \int_a^{a+2\pi} f(x) dx$$

$$(c) \int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} f(c+x) dx, \text{ where } a, b, c \text{ are any numbers.}$$

5.4. Trigonometric Series

If a set of trigonometric functions $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx, \dots\}$ is expressed in the form as $a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots + a_n \cos nx + b_n \sin nx + \dots$ then it is called a trigonometric series where $a_0, a_1, a_2, \dots, a_n, \dots, b_1, b_2, \dots, b_n, \dots$ are the real constants and are known as the coefficients of the trigonometric series. And the set $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx, \dots\}$ is called a trigonometric system.

5.5. Orthogonality of Two Functions

Two functions $f(x)$ and $g(x)$ are called orthogonal over an interval $[a, b]$ if $\int_a^b f(x) g(x) dx = 0$.

Similarly a trigonometric system $1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx, \dots$ is called orthogonal in an interval $[-\pi, \pi]$ if

$$(a) \int_{-\pi}^{\pi} \sin mx \sin nx dx = 0 \text{ for } m \neq n$$

$$(b) \int_{-\pi}^{\pi} \cos mx \cos nx dx = 0 \text{ for } m \neq n \text{ and}$$

$$(c) \int_{-\pi}^{\pi} \sin mx \cos nx dx = 0 \text{ for any integers } m \text{ and } n \text{ including } m = n.$$

Note:

The above trigonometric system is also orthogonal in any interval of length 2π .

5.6. Fourier Series

If the trigonometric series $a_0 + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx + \dots + b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx + \dots = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$ uniformly converges to $f(x)$ in any interval $I \in R$, then

the series is called the Fourier series where a_0, a_n, b_n are the constants to be determined and are known as the Fourier coefficients.

5.7. Fourier Series of a Function having Period 2π

The Fourier series a periodic function $f(x)$ in an interval $(-\pi, \pi)$ is defined as the sum of the trigonometric series $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$ which is uniformly converges to $f(x)$. It can be expressed $f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$ where the Fourier coefficients a_0, a_1, b_n can be determined by the Euler's formulae:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \text{ and}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \text{ where } n = 1, 2, 3, \dots$$

5.8. Determination of Fourier Coefficients

The Fourier series of a periodic function $f(x)$ in an interval $(-\pi, \pi)$ is defined by

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad (1)$$

(a) To determine a_0

To find a_0 integrating (1) with respect to 'x' on both sides term by term from $-\pi$ to π then we have,

$$\int_{-\pi}^{\pi} f(x) dx = \frac{1}{2} \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos nx dx + b_n \int_{-\pi}^{\pi} \sin nx dx \right]$$

$$\text{or, } \int_{-\pi}^{\pi} f(x) dx = \frac{a_0}{2} [x]_{-\pi}^{\pi} + \sum_{n=1}^{\infty} \left[a_n \frac{\sin nx}{n} \Big|_{-\pi}^{\pi} + b_n \left(-\frac{\cos nx}{n} \right) \Big|_{-\pi}^{\pi} \right]$$

$$\text{or, } \int_{-\pi}^{\pi} f(x) dx = \frac{a_0}{2} [\pi + \pi] + \sum_{n=1}^{\infty} \left[a_n \left(\frac{\sin n\pi}{n} + \frac{\sin n\pi}{n} \right) - b_n \left(\frac{\cos n\pi}{n} - \frac{\cos n\pi}{n} \right) \right]$$

$$\text{or, } \int_{-\pi}^{\pi} f(x) dx = \frac{a_0}{2} [2\pi] + \sum_{n=1}^{\infty} [a_n (0 + 0) - b_n (0)] \quad [\because \sin n\pi = 0]$$

$$\text{or, } \int_{-\pi}^{\pi} f(x) dx = a_0 \pi + 0$$

$$\text{or, } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$\therefore a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

(b) To determine a_n

To find a_n multiplying (1) by $\cos nx$ and integrating with respect to 'x' on both sides term by term from $-\pi$ to π we have,

$$\int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{2} a_0 \int_{-\pi}^{\pi} \cos nx dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos^2 nx dx + b_n \int_{-\pi}^{\pi} \sin nx \cos nx dx \right]$$

$$\text{or, } \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{2} a_0 \left[\frac{\sin nx}{n} \right]_{-\pi}^{\pi} + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \frac{1 + \cos 2nx}{2} dx + \frac{b_n}{2} \int_{-\pi}^{\pi} \sin 2nx dx \right]$$

$$\begin{aligned}
&= \frac{1}{2} a_0 [0] + \sum_{n=1}^{\infty} \left[\frac{a_n}{2} \left(x + \frac{\sin 2nx}{2n} \right) \Big|_{-\pi}^{\pi} + \frac{b_n}{2} \left(-\frac{\cos 2nx}{2n} \right) \Big|_{-\pi}^{\pi} \right] \\
&= 0 + \sum_{n=1}^{\infty} \left[\frac{a_n}{2} (\pi + \pi + 0) - \frac{b_n}{4n} (\cos 2n\pi - \cos 2n\pi) \right] \quad [\because \sin n\pi = 0]
\end{aligned}$$

$$\text{or, } \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \sum_{n=1}^{\infty} \left[\frac{a_n}{2} (2\pi) - 0 \right] = a_n \pi$$

$$\therefore a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \text{ where, } n = 1, 2, 3, \dots$$

(c) **To determine b_n**

To find b_n multiplying (1) $\sin nx$ and integrating with respect to 'x' on both sides term by term from $-\pi$ to π we have,

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x) \sin nx \, dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} \sin nx \, dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \sin nx \cos nx \, dx + b_n \int_{-\pi}^{\pi} \sin^2 nx \, dx \right] \\
&= \frac{a_0}{2} \left[-\frac{\cos nx}{n} \right]_{-\pi}^{\pi} + \sum_{n=1}^{\infty} \left[\frac{a_n}{2} \int_{-\pi}^{\pi} \sin 2nx \, dx + b_n \int_{-\pi}^{\pi} \frac{1 - \cos 2nx}{2} \, dx \right] \\
&= -\frac{a_0}{2n} [\cos n\pi - \cos n\pi] + \sum_{n=1}^{\infty} \left[\frac{a_n}{2} \left(-\frac{\cos 2nx}{2n} \right) \Big|_{-\pi}^{\pi} + \frac{b_n}{2} \left(x - \frac{\sin 2nx}{2n} \right) \Big|_{-\pi}^{\pi} \right] \\
&= 0 + \sum_{n=1}^{\infty} \left[0 + \frac{b_n}{2} (2\pi) \right] = \pi b_n
\end{aligned}$$

$$\text{or, } \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \pi b_n$$

$$\therefore b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \text{ where } n = 1, 2, 3, \dots$$

Note: The coefficients a_0, a_n, b_n are called Euler's coefficients.

Example: 2

Find the Fourier series expansion for $f(x) = \pi + x$ in $(-\pi, \pi)$.

Solution:

Let the Fourier series of the function $f(x)$ in $(-\pi, \pi)$ is defined by

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$\text{That is, } \pi + x = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad (1)$$

$$\begin{aligned}
 \text{where } a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi + x) dx = \frac{1}{\pi} \left[\pi x + \frac{x^2}{2} \right]_{-\pi}^{\pi} \\
 &= \frac{1}{\pi} \left[\pi(\pi) + \frac{\pi^2}{2} - \left\{ \pi(-\pi) + \frac{(-\pi)^2}{2} \right\} \right] \\
 &= \frac{1}{\pi} \left[\pi^2 + \frac{\pi^2}{2} + \pi^2 - \frac{\pi^2}{2} \right] \\
 &= \frac{1}{\pi} [2\pi^2] = 2\pi
 \end{aligned}$$

$$\therefore a_0 = 2\pi$$

$$\text{and, } a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi + x) \cos nx dx$$

$$= \frac{1}{\pi} \left[(\pi + x) \frac{\sin nx}{n} - (0 + 1) \left(-\frac{\cos nx}{n^2} \right) \right]_{-\pi}^{\pi}$$

[$\because uv dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$ where dashes indicate the differentiation and that of suffices for integration]

$$\begin{aligned}
 &= \frac{1}{\pi} \left[(\pi + x) \frac{\sin nx}{n} + \frac{\cos nx}{n^2} \right]_{-\pi}^{\pi} \\
 &= \frac{1}{\pi} \left[(\pi + \pi) \frac{\sin n\pi}{n} + \frac{\cos n\pi}{n^2} - (\pi - \pi) \frac{\sin(-\pi)}{n} - \frac{\cos n(-\pi)}{n^2} \right] \\
 &= \frac{1}{\pi} \left[2\pi(0) + \frac{(-1)^n}{n^2} - 0 - \frac{(-1)^n}{n^2} \right] = \frac{1}{\pi} [0] \quad [\because \cos n\pi = (-1)^n]
 \end{aligned}$$

$$\therefore a_n = 0$$

$$\text{Again, } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi + x) \sin nx dx$$

$$\begin{aligned}
 &= \frac{1}{\pi} \left[(\pi + x) \left(-\frac{\cos nx}{n} \right) - (0 + 1) \left(-\frac{\sin nx}{n^2} \right) \right]_{-\pi}^{\pi} \\
 &= \frac{1}{\pi} \left[-(\pi + x) \frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right]_{-\pi}^{\pi} \\
 &= \frac{1}{\pi} \left[-(\pi + \pi) \frac{\cos n\pi}{n} + \frac{\sin n\pi}{n^2} + (\pi - \pi) \frac{\cos n(-\pi)}{n} - \frac{\sin n(-\pi)}{n^2} \right] \\
 &= \frac{1}{\pi} \left[-2\pi \frac{(-1)^n}{n} + 0 + 0 + 0 \right] = \frac{1}{\pi} \left[2\pi \frac{(-1)^{n+1}}{n} \right]
 \end{aligned}$$

$$\therefore b_n = \frac{2}{n} (-1)^{n+1}$$

For the values of a_0 , a_n and b_n the equation (1) becomes as,

$$\pi + x = \frac{1}{2}(2\pi) + \sum_{n=1}^{\infty} \left[(0) \cos nx + \frac{2(-1)^{n+1}}{n} \sin nx \right]$$

$$\therefore \pi - x = \pi + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

Example: 3

Find the Fourier series of $f(x)$ in the interval $(-\pi, \pi)$ defined by

$$f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 \leq x < \pi/2 \\ 0 & \text{for } \pi/2 \leq x \leq \pi \end{cases}$$

Solution:

Let the Fourier series of $f(x)$ in $(-\pi, \pi)$ is defined by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad (1)$$

$$\text{where, } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^0 (0) dx + \frac{1}{\pi} \int_0^{\pi/2} (1) dx + \frac{1}{\pi} \int_{\pi/2}^{\pi} (0) dx$$

$$= \frac{1}{\pi} [0 + x]_0^{\pi/2} + 0$$

$$= \frac{1}{\pi} [x]_0^{\pi/2} = \frac{1}{\pi} [\pi/2 - 0]$$

$$\therefore a_0 = \frac{1}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 (0) \cos nx dx + \int_0^{\pi/2} (1) \cos nx dx + \int_{\pi/2}^{\pi} 0 \cdot \cos nx dx \right]$$

$$\text{or, } a_n = \frac{1}{\pi} \left[0 + \int_0^{\pi/2} \cos nx dx + 0 \right]$$

$$= \frac{1}{\pi} \left[\frac{\sin nx}{n} \right]_0^{\pi/2} = \frac{1}{n\pi} \left[\cos n \frac{\pi}{2} - 1 \right]$$

$$\therefore b_n = \frac{1}{n\pi} \left[1 - \cos n \frac{\pi}{2} \right]$$

Now, for the values of a_0 , a_n and b_n the equation (1) takes the form as,

$$\begin{aligned}
f(x) &= \frac{1}{2} \left(\frac{1}{2} \right) + \sum_{n=1}^{\infty} \left[\frac{1}{n\pi} \sin \frac{n\pi}{2} \cos nx + \frac{1}{n\pi} \left(1 - \cos \frac{n\pi}{2} \right) \sin nx \right] \\
&= \frac{1}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\sin \frac{n\pi}{2} \cdot \frac{\cos nx}{n} + \left(1 - \cos \frac{n\pi}{2} \right) \frac{\sin nx}{n} \right] \\
&= \frac{1}{4} + \frac{1}{\pi} \left[(1) \frac{\cos x}{1} + 0 + (-1) \frac{\cos 3x}{3} + 0 + (+1) \frac{\cos 5x}{5} + \dots + (1-0) \frac{\sin x}{1} + (1+1) \right. \\
&\quad \left. \frac{\sin 2x}{2} + (1-0) \frac{\sin 3x}{3} + (1-1) \frac{\sin 4x}{4} + (1-0) \frac{\sin 5x}{5} + \dots \right] \\
&= \frac{1}{4} + \frac{1}{\pi} \left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \dots + \sin x + \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right] \\
\therefore f(x) &= \frac{1}{4} + \frac{1}{\pi} \left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \dots + \sin x + \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right]
\end{aligned}$$

5.9. The Fourier Series of $f(x)$ in the Interval $(0, 2\pi)$

The Fourier series of a periodic function $f(x)$ in $(0, 2\pi)$ is defined by

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \text{ where, } a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx, n = 1, 2, 3, \dots \text{ and } b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx, n = 1, 2, 3, \dots$$

Example: 4

Find the Fourier series of $f(x)$ on the given interval defined by

$$f(x) = \begin{cases} 0 & \text{for } 0 < x < \pi \\ 1 & \text{for } \pi < x < 2\pi \end{cases}$$

Solution:

Let the Fourier series of the periodic function $f(x)$ in $(0, 2\pi)$ is

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad (1)$$

$$\text{where, } a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} (0) dx + \int_{\pi}^{2\pi} (1) dx \right] = \frac{1}{\pi} [0 + x]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi} [2\pi - \pi] = \frac{1}{\pi} [\pi] = 1$$

$$\therefore a_0 = 1$$

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\
 &= \frac{1}{\pi} \left[\int_0^{\pi} (0) \cos nx \, dx + \int_{\pi}^{2\pi} (1) \cos nx \, dx \right] \\
 &= \frac{1}{\pi} \left[0 + \frac{\sin nx}{n} \Big|_{\pi}^{2\pi} \right] \\
 &= \frac{1}{\pi} \left[\frac{\sin n(2\pi)}{n} - \frac{\sin n\pi}{n} \right] \\
 &= \frac{1}{\pi} [0 - 0] = \frac{1}{\pi} [0]
 \end{aligned}$$

$$\therefore a_n = 0$$

$$\begin{aligned}
 \text{Again, } b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx \\
 &= \frac{1}{\pi} \left[\int_0^{\pi} (0) \sin nx \, dx + \int_{\pi}^{2\pi} (1) \sin nx \, dx \right] \\
 &= \frac{1}{\pi} \left[0 + \frac{(-\cos nx)}{n} \Big|_{\pi}^{2\pi} \right] \\
 &= -\frac{1}{n\pi} [\cos 2n\pi - \cos n\pi] \\
 &= -\frac{1}{n\pi} [(-1)^{2n} - (-1)^n] \\
 &= -\frac{1}{n\pi} [1 + (-1)^{n+1}]
 \end{aligned}$$

$$\therefore b_n = -\frac{1}{\pi} \left[\frac{1 + (-1)^{n+1}}{n} \right]$$

Now, substituting the values of a_0 , a_n and b_n in (1) we get,

$$f(x) = \frac{1}{2} (1) + \sum_{n=1}^{\infty} \left[0 - \frac{1}{\pi} \left\{ \frac{1 + (-1)^{n+1}}{n} \right\} \sin nx \right]$$

$$\begin{aligned}
 \text{or, } f(x) &= \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} + 1}{n} \sin nx \\
 &= \frac{1}{2} - \frac{2}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right)
 \end{aligned}$$

$$\therefore f(x) = \frac{1}{2} - \frac{2}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right)$$

which is the required Fourier series.

5.10. Even and Odd Functions

- (a) A function $f(x)$ is said to be an even if $f(-x) = f(x)$ for all x . Such as, x^2 , $\cos x$, $\sec x$, $|\sin x|$, $|x|$, $\sin^2 x$, $\sin^2 x \cos x$ are all even functions. For example,

$$\text{If } f(x) = x^2 \text{ then}$$

$$\Rightarrow f(-x) = (-x)^2 = x^2 = f(x)$$

$$\therefore f(-x) = f(x)$$

$$f(x) = x^2 \text{ is an even function.}$$

Similarly, we can test for other also.

- (b) A function $f(x)$ is said to be an odd if $f(-x) = -f(x)$ for all x . Such as, x , x^3 , $\sin x$, $\tan x + \sin x$ etc. are examples of the odd functions. For example,

$$\text{If } f(x) = \sin x \text{ then}$$

$$f(-x) = \sin(-x) = -\sin x = -f(x)$$

$$\Rightarrow f(-x) = -f(x)$$

$$\therefore f(x) = \sin x \text{ is an odd function.}$$

Similarly, we can test for others.

Note: The graph of even function will be symmetrical with respect to the y -axis and that of odd functions will be symmetrical with respect to the origin.

Properties of even and odd functions

(a) If $f(x)$ is even function, then $\int_{-P}^P f(x) dx = 2 \int_0^P f(x) dx$

(b) If $f(x)$ is odd function then $\int_{-P}^P f(x) dx = 0$.

Example: 5

Check whether $f(x) = x^2 \sin^2 x + \cos x$ is even or odd.

Solution:

$$\text{Here, } f(x) = x^2 \sin^2 x + \cos x$$

$$\begin{aligned} \text{or, } f(-x) &= (-x)^2 \sin^2(-x) + \cos(-x) \\ &= x^2 (-\sin x)^2 + \cos x \\ &= x^2 \sin^2 x + \cos x = f(x). \end{aligned}$$

Since $f(-x) = f(x)$ so the given function is even.

Now, let the Fourier series of the given function $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad (1)$$

where, $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$

Since $f(x)$ is even, we have

$$\begin{aligned} a_0 &= \frac{1}{\pi} 2 \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} 2x dx \\ &= \frac{4}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \frac{2}{\pi} (\pi^2 - 0) = 2\pi \end{aligned}$$

$\therefore a_0 = 2\pi$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \cos nx \quad [\text{an even function}] \\ &= \frac{1}{\pi} \left[\int_{-\pi}^0 (-2x) \cos nx dx + \int_0^{\pi} (2x) \cos nx dx \right] \\ &= -\frac{2}{\pi} \int_{-\pi}^0 x \cos nx dx + \frac{2}{\pi} \int_0^{\pi} x \cos nx dx \\ &= -\frac{2}{\pi} \left[x \frac{\sin nx}{n} - (1) \left(\frac{-\cos nx}{n^2} \right) \right]_{-\pi}^0 + \frac{2}{\pi} \left[x \frac{\sin nx}{n} - (1) \left(\frac{-\cos nx}{n^2} \right) \right]_0^{\pi} \\ &= -\frac{2}{\pi} \left[-0 + \frac{1}{n^2} - 0 - \frac{(-1)^n}{n^2} \right] + \frac{2}{\pi} \left[0 + \frac{(-1)^n}{n^2} - 0 - \frac{1}{n^2} \right] \\ &= -\frac{2}{\pi} \left[\frac{1}{n^2} - \frac{(-1)^n}{n^2} - \frac{(-1)^n}{n^2} + \frac{1}{n^2} \right] \\ &= -\frac{2}{\pi} \left[\frac{2}{n^2} - \frac{(-1)^n}{n^2} \right] = -\frac{4}{\pi} \left[\frac{1 - (-1)^n}{n^2} \right] \end{aligned}$$

$\therefore a_n = \frac{4}{\pi} \left[\frac{1 - (-1)^n}{n^2} \right]$

Again, $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = 0$ [$\because f(x) \sin nx$ is odd]

Putting the values of a_0 , a_n and b_n in (1) we get,

$$f(x) = \frac{1}{2} (2\pi) + \sum_{n=1}^{\infty} \left[\frac{4}{\pi} \frac{(-1)^n - 1}{n^2} \cos nx + 0 \right] \quad [\because b_n = 0]$$

$\therefore f(x) = \pi + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx$

5.11. The Fourier Series in terms of the Even and the Odd Functions

The Fourier series of a function $f(x)$ in the interval $(-\pi, \pi)$ is defined by

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad (1)$$

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If $f(x)$ be an even function, then $f(x) \cos x$ will be even and $f(x) \sin x$ be the odd functions. Thus we have,

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{2}{\pi} \int_0^{\pi} f(x) dx \end{aligned} \quad [\text{By the property of even function}]$$

$$\begin{aligned} \text{or, } a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\ &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx \end{aligned} \quad [\text{By property of even function}]$$

$$\text{and } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = 0 \quad [\text{By property of odd function}]$$

Now, the equation (1) reduces as,

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad [\because b_n = 0] \quad (2)$$

The relation (2) is known as the Fourier cosine series of $f(x)$ in $(0, \pi)$ where the Fourier coefficients a_0 and a_n can be obtained by the Euler's formulae as below,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx \text{ and } a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

Case II

If $f(x)$ be an odd function, then $f(x) \cos x$ will be odd and $f(x) \sin x$ be an even functions respectively. Thus we have,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = 0 \quad [\because \text{Because of odd function}]$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = 0 \quad [\because \text{Odd function}]$$

$$\begin{aligned} \text{And } b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\ &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx \end{aligned} \quad [\because \text{Because of even function}]$$

Now, the equation (1) reduces as,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad [\because a_0 = a_n = 0] \quad (3)$$

The relation (3) is known as the Fourier sine series of $f(x)$ in $(0, \pi)$ where the Fourier coefficient b_n can be obtained by the Euler's formula as,

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

Notes:

- (i) If $f(x)$ is even then $b_n = 0$ and find the Fourier cosine series.
- (ii) If $f(x)$ is odd then $a_0 = a_n = 0$ and have to find the Fourier sine series.
- (iii) The product of two even (or odd) functions is again even.
- (iv) The product of even and odd functions is odd.
- (v) The sum (or difference) of two even functions is even.
- (vi) The sum (or difference) of two odd functions is odd.

Example: 7

Obtain the Fourier series for $f(x) = x^3$ in $(-\pi, \pi)$.

Solution:

Let the Fourier series of $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad (1)$$

$$\text{where, } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^3 \, dx = 0 \quad [\because f(x) = x^3 \text{ is an odd}]$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^3 \cos nx \, dx = 0 \quad [\because x^3 \cos nx \text{ is an odd}]$$

$$\text{Again, } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^3 \sin nx \, dx$$

$$\text{or, } b_n = \frac{2}{\pi} \int_0^{\pi} x^3 \sin nx \, dx \quad [\because x^3 \sin nx \text{ is even function}]$$

$$\begin{aligned} &= \frac{2}{\pi} \left[x^3 \left(-\frac{\cos nx}{n} \right) - 3x^2 \left(-\frac{\sin nx}{n^2} \right) + 6x \left(\frac{\cos nx}{n^3} \right) - 6 \left(\frac{\sin nx}{n^4} \right) \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[-\frac{x^3 \cos nx}{n} + \frac{3}{n^2} x^2 \sin nx + \frac{6}{n^3} x \cos nx - \frac{6}{n^4} \sin nx \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[-\frac{\pi^3 \cos n\pi}{n} + \frac{3}{n^2} \pi^2 \sin n\pi + \frac{6}{n^3} \pi \cos n\pi - \frac{6}{n^4} \sin n\pi - (0 + 0 + 0 + 0) \right] \end{aligned}$$

$$= \frac{2}{\pi} \left[-\frac{\pi^3 (-1)^n}{n} + 0 + \frac{6\pi}{n^3} (-1)^n - 0 - 0 \right]$$

$$= 2(-1)^n \left[\frac{6}{n^3} - \frac{\pi^2}{n} \right]$$

$$\therefore b_n = 2(-1)^n \left[\frac{6}{n^3} - \frac{\pi^2}{n} \right]$$

Now, the equation (1) takes the form for the values of a_0 , a_n and b_n as

$$\text{i.e. } x^3 = 0 + \sum_{n=1}^{\infty} \left[0 + 2(-1)^n \left(\frac{6}{n^3} - \frac{\pi^2}{n} \right) \sin nx \right]$$

$$\therefore x^3 = 2 \sum_{n=1}^{\infty} (-1)^n \left(\frac{6}{n^3} - \frac{\pi^2}{n} \right) \sin nx$$

Example: 8

Expand the function $f(x) = x \sin x$ as a Fourier series in $[-\pi, \pi]$.

Solution:

Let the Fourier series of $f(x)$ in $[-\pi, \pi]$ is defined by

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad (1)$$

Since $f(x)$ is even function in $[-\pi, \pi]$. So $b_n = 0$.

The equation (1) becomes as,

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad [\because b_n = 0] \quad (2)$$

$$\text{where, } a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x \sin x dx$$

$$\text{or, } a_0 = \frac{2}{\pi} [x(-\cos x) - (1)(-\sin x)]_0^{\pi}$$

$$= \frac{2}{\pi} [-x \cos x + \sin x]_0^{\pi}$$

$$\text{or, } a_0 = \frac{2}{\pi} [-\pi \cos \pi + \sin \pi + 0 - 0]$$

$$= -\frac{2}{\pi} [\pi(-1) + 0] = 2$$

$$\therefore a_0 = 2$$

$$\text{And } a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin x \cdot \cos nx \, dx \quad (3)$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \cdot 2 \sin x \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x [\sin (n+1)x - \sin (n-1)x] \, dx$$

$$[\because 2 \cos A \sin B = \sin (A+B) - \sin (A-B)]$$

$$= \frac{1}{\pi} \int_0^{\pi} x \sin (n+1)x \, dx - \frac{1}{\pi} \int_0^{\pi} x \sin (n-1)x \, dx$$

$$= \frac{1}{\pi} \left[x \left\{ -\frac{\cos (n+1)x}{(n+1)} \right\} + \frac{\sin (n+1)x}{(n+1)^2} \right]_0^{\pi} - \left[x \left\{ -\frac{\cos (n-1)x}{(n-1)} \right\} + \frac{\sin (n-1)x}{(n-1)^2} \right]_0^{\pi}$$

$$(n \neq 1)$$

$$= \frac{1}{\pi} \left[-\frac{\pi(-1)^{n+1}}{n+1} + 0 + 0 - 0 \right] - \frac{1}{\pi} \left[-\frac{\pi(-1)^{n-1}}{n-1} + 0 + 0 - 0 \right]$$

$$= \frac{1}{\pi} \left[-\frac{\pi(-1)^{n+1}}{n+1} \right] - \frac{1}{\pi} \left[-\frac{\pi(-1)^{n-1}}{n-1} \right] \quad (n \neq 1)$$

$$= \frac{\pi}{\pi} \left[\frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} \right] \quad (n \neq 1)$$

$$= \frac{(-1)^n [n-1-n-1]}{(n+1)(n-1)} = \frac{-2(-1)^n}{(n+1)(n-1)} \quad (n \neq 0)$$

$$\therefore a_n = \frac{-2(-1)^n}{(n+1)(n-1)} (n \neq 1) = \frac{-2(-1)^n}{n^2-1}$$

For $n = 1$, from (3) we have

$$a_1 = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos x \, dx$$

$$\text{or, } a_1 = \frac{1}{\pi} \int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi} \left[x \left(-\frac{\cos 2x}{2} \right) + (1) \frac{\sin 2x}{4} \right]_0^{\pi}$$

$$\text{or, } a_1 = \frac{1}{\pi} \left[-\pi \cdot \frac{\cos 2\pi}{2} + \frac{\sin 2\pi}{4} - 0 - 0 \right]$$

$$\text{or, } a_1 = \frac{1}{\pi} \left[-\pi \cdot \frac{1}{2} + 0 \right] = -\frac{1}{2}$$

$$\therefore a_1 = -\frac{1}{2}$$

For the values of a_0 , a_1 and a_n the equation (2) becomes as,

$$f(x) = \frac{1}{2}(2) - \frac{1}{2} \cos x - 2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 - 1} \cos nx$$

$$\therefore x \sin x = 1 - \frac{1}{2} \cos x - 2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 - 1} \cos nx$$

Example: 9

Find the Fourier series of a periodic function $f(x)$ with period 2π which generates the rectangular wave defined by

$$f(x) = \begin{cases} -3 & \text{for } -\pi < x < 0 \\ 3 & \text{for } 0 < x < \pi \end{cases}$$

Solution:

Let the Fourier series of the rectangular wave function $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad (1)$$

$$\text{where } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^0 (-3) dx + \frac{1}{\pi} \int_0^{\pi} (3) dx$$

$$= \frac{-3}{\pi} [x]_{-\pi}^0 + \frac{3}{\pi} [x]_0^{\pi} = \frac{-3}{\pi} [0 + \pi] + \frac{3}{\pi} [\pi - 0]$$

$$= -3 + 3 = 0$$

$$\therefore a_0 = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^0 (-3) \cos nx dx + \frac{1}{\pi} \int_0^{\pi} (3) \cos nx dx$$

$$= \frac{-3}{\pi} \left[\frac{\sin nx}{n} \right]_{-\pi}^0 + \frac{3}{\pi} \left[\frac{\sin nx}{n} \right]_0^{\pi}$$

$$= \frac{-3}{n\pi} [0 + 0 + 0 - 0] = 0$$

$$\therefore a_n = 0$$

$$\text{Again, } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 (-3) \sin nx dx + \int_0^{\pi} (3) \sin nx dx \right]$$

$$= \frac{3}{\pi} \left[-\left(\frac{\cos nx}{n} \right) \Big|_{-\pi}^0 + \left(\frac{\cos nx}{n} \right) \Big|_0^{\pi} \right]$$

$$= \frac{3}{n\pi} [2 - (-1)^n - (-1)^n]$$

$$= \frac{3}{n\pi} [2 - 2(-1)^n]$$

$$= \frac{6}{n\pi} [1 - (-1)^n]$$

$$\therefore b_n = \frac{6}{n\pi} [1 - (-1)^n]$$

Now, for the values of a_0 , a_n and b_n the equation (1) becomes as,

$$f(x) = 0 + \sum_{n=1}^{\infty} \left[0 + \frac{6}{n\pi} \{1 - (-1)^n\} \sin nx \right]$$

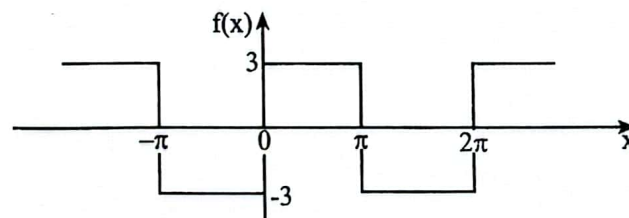
$$\text{or, } f(x) = \sum_{n=1}^{\infty} \frac{6}{n\pi} [1 - (-1)^n] \sin nx$$

$$\therefore f(x) = \frac{6}{\pi} \sum_{n=1}^{\infty} \left[\frac{1 - (-1)^n}{n} \right] \sin nx$$

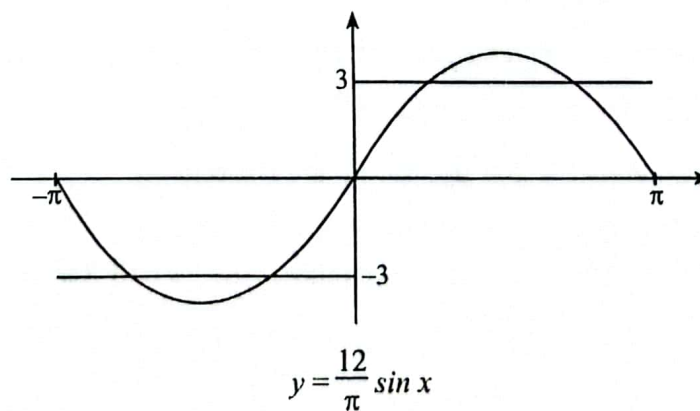
$$= \frac{6}{\pi} \sum_{n=1}^{\infty} \left[\frac{2}{1} \sin x + \frac{2}{3} \sin 3x + \frac{2}{5} \sin 5x + \dots \right]$$

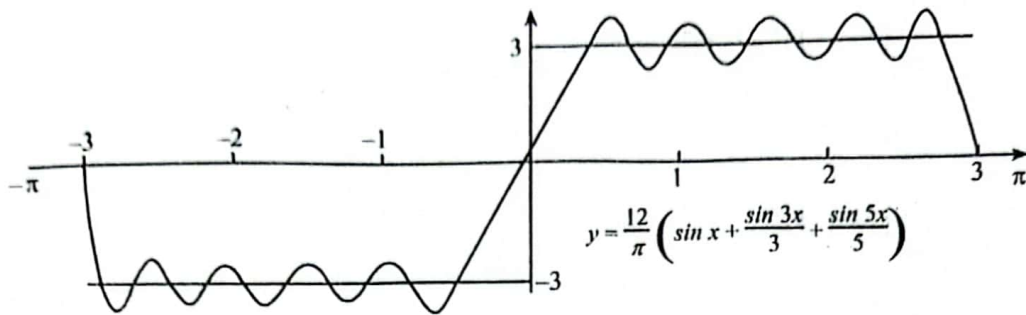
$$= \frac{12}{\pi} \sum_{n=1}^{\infty} \left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right]$$

The graph of some partial sums of the given rectangular wave are as below:



Periodic square wave of the given function $f(x)$





The graph of the first three partial sums of the given function $f(x)$ which possesses a periodic square wave.

Note: A periodic function $f(x)$ defined by

$$f(x) = \begin{cases} -K & \text{for } -\pi < x < 0 \\ K & \text{for } 0 < x < \pi \end{cases} \text{ with period } 2\pi$$

generates a rectangular wave in $(-\pi, \pi)$ which occur as external forces acting on mechanical system, electromotive forces in electric circuit, and etc.

EXERCISE - 5(I)

Difficulty Level: A

- Define a periodic function. Write the smallest periods of (a) $\sin \theta$ (b) $\tan \theta$ (c) $\cos 2\theta$
- Write three properties of periodic functions.
- Write the Fourier series of a periodic function $f(x)$ in an interval $(-\pi, \pi)$ and also write the Euler's formulae to find the Fourier coefficients a_0 , a_n and b_n .
- Define even and odd functions with at least one example for each.
- What are the properties of even and odd functions? Write them.
- Determine the Fourier coefficients (a) a_0 (b) a_n (c) b_n to each of the following periodic function $f(x)$ defined by
 - $f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 \leq x < \pi \end{cases}$
 - $f(x) = x^2$ within the interval $(0, 2\pi)$
 - $f(x) = \begin{cases} 1 & \text{for } -\pi < x < 0 \\ -1 & \text{for } 0 \leq x < \pi \end{cases}$
- If the values of Fourier coefficients respectively $a_0 = \frac{3}{2}$, $a_n = \frac{2}{n^2\pi^2} [(-1)^n - 1]$ and $b_n = \frac{-2}{n\pi}$ then form the Fourier series for the function $f(x)$ defined in $[-2, 2]$.
- If the values of Fourier coefficients a_0 , a_n , b_n are respectively $\frac{2\pi^2}{3}$, $\pm \frac{4}{n^2}$ and 0 then determine the Fourier series of the function defined by $f(x) = x^2$ in $(-\pi, \pi)$
- If a periodic function $f(x)$ in $(-\pi, \pi)$ is an even then write its Half Range Fourier cosine series and that of Fourier Coefficients a_0 and a_n .
- If a periodic function $f(x)$ in $(-\pi, \pi)$ is an odd function then write the Half Range Fourier sine series and that of Fourier coefficient b_n .
- Define about the fundamental period of a periodic function. Find the fundamental period of the following periodic functions.

- (a) $\sin \pi x$ (b) $\cos 2\pi x$ (c) $\tan 3x$
 (d) $\cot nx$ (e) $\sin 5x$ (f) $\cos \frac{2\pi x}{k}$
 (g) $\sin \frac{2n\pi x}{k}$

12. Identify whether the following functions are even, odd or neither.

- (a) x^7 (b) $x + x^2$ (c) $|x|$
 (d) $x \sin x$ (e) $x \cos nx$ (f) $|\sin x|$
 (g) $x|x|$ (h) $\sin^2 x$ (i) $\log x$
 (j) $e^x + e^{-x}$ (k) $e^x - e^{-x}$

Difficulty Level: B

13. Obtain the Fourier series of $f(x)$ in the given interval

(a) $f(x) = \begin{cases} 1 & \text{for } -\pi < x < 0 \\ -1 & \text{for } 0 \leq x < \pi \end{cases}$

(b) $f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 \leq x < \pi \end{cases}$

(c) $f(x) = \begin{cases} -1 & \text{if } 0 < x < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} < x < 2\pi \end{cases}$

14. Obtain the Fourier series of $f(x)$ defined by

$$f(x) = \begin{cases} -1 & \text{for } -\pi < x < -\frac{\pi}{2} \\ 0 & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 1 & \text{for } \frac{\pi}{2} < x < \pi \end{cases}$$

15. Obtain the Fourier series of $f(x)$ in $(-\pi, \pi)$ defined by

(a) $f(x) = -5x$ (b) $f(x) = x^2$

16. Find the Fourier series of $f(x)$ in $(0, 2\pi)$.

(a) $f(x) = x$ (b) $f(x) = \pi - x$ (c) $f(x) = 2x^2$
 (d) $f(x) = -x^2$ (e) $f(x) = x^2$

17. Find the Fourier series of $f(x)$ defined by

$$f(x) = \begin{cases} 0 & \text{for } -\pi \leq x \leq 0 \\ \frac{\pi}{4}x & \text{for } 0 < x \leq \pi \end{cases}$$

18. Find the Fourier series of $f(x) = \begin{cases} x & \text{for } 0 \leq x \leq \pi \\ 2\pi - x & \text{for } \pi < x \leq 2\pi \end{cases}$

19. Test whether the following functions are even or odd. And hence find the Fourier series of the functions where,

(a) $f(x) = x^2$ in $[-\pi, \pi]$ (b) $f(x) = |x|$ in $(-\pi, \pi)$

(c) $f(x) = \begin{cases} 0 & \text{for } -\pi \leq x < -\frac{\pi}{2} \\ x & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} < x \leq \pi \end{cases}$

20. Express $f(x)$ as a Fourier series in $[-\pi, \pi]$ which is defined by

$$f(x) = \begin{cases} -\pi - x & \text{for } -\pi \leq x \leq 0 \\ \pi + x & \text{for } 0 < x \leq \pi \end{cases}$$

ANSWERS

1. (a) 2π (b) π (c) π

3. $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$, Where $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$ and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \text{ where } n = 1, 2, 3, \dots$$

6. (a) (i) $a_0 = \frac{1}{2}$, (ii) $a_n = 0$, (iii) $b_n = \frac{1}{\pi} \left[\frac{1 - (-1)^n}{n} \right]$

(b) (i) $a_0 = \frac{4\pi^2}{2}$, (ii) $a_n = \frac{4}{n^2}$, (iii) $b_n = -\frac{4\pi}{n}$

(c) (i) $a_0 = 0$, (ii) $a_n = 0$, (iii) $b_n = \frac{(-1)^n - 2}{n\pi}$

7. $f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \left[\frac{2}{n^2\pi^2} \{(-1)^n - 1\} \cos \frac{n\pi x}{2} - \frac{2}{n\pi} \sin \frac{n\pi x}{2} \right]$

8. $x^2 = \frac{\pi^2}{4} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$

9. $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) \, dx$, $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$,

10. $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ where $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$,

11. (a) 2 (b) 1 (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{n}$

(e) $\frac{2\pi}{5}$ (f) k (g) $\frac{k}{n}$

12. (a) Odd (b) Neither (c) Even
 (d) Even (e) Odd (f) Even
 (g) Odd (h) Even (i) Neither
 (j) Even (k) Odd

13. (a) $x - \frac{4}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right)$

(b) $\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n\pi} \sin nx$

(c) $-\frac{1}{4} - \frac{1}{\pi} \left(\cos x - \cos 3x \frac{1}{3} + \cos 5x \frac{1}{5} + \dots + \frac{\sin x}{1} + \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \dots \right)$

14. $\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$

15. (a) $10 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$

(b) $\frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \left[\frac{2(-1)^n}{n^2} \cos nx \right]$

16. (a) $\pi - 2 \left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{n} + \dots \right)$

(b) $2 \left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{n} + \dots \right)$

(c) $\frac{8\pi^2}{3} + 8 \sum_{n=1}^{\infty} \left(\frac{\cos nx}{n^2} - \frac{\pi \sin nx}{n} \right)$

(d) $\frac{-4\pi^2}{3} - 4 \sum_{n=1}^{\infty} \left(\frac{\cos nx}{n^2} - \frac{\pi \sin nx}{n} \right)$

(e) $\frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \left(\frac{1}{n^2} \cos nx - \frac{\pi \sin nx}{n} \right)$

17. $\frac{\pi^2}{16} + \frac{1}{4} \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2} \cos nx - \frac{(-1)^n}{n} \sin nx \right]$

18. $\frac{\pi}{2} + 2 \sum_{n=1}^{\infty} \left[\frac{-1 + (-1)^n}{n^2} \right] \cos nx$

19. (a) Even, $\frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx$

(b) Even, $\frac{\pi}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx$

(c) Odd, $\frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{\sin \frac{n\pi}{2}}{n^2} - \frac{\pi \cos \frac{n\pi}{2}}{2n} \right] \sin nx$

20. $\frac{\pi}{2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2} \cos nx + \frac{1 - (-1)^n}{n} \sin nx \right]$

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Model Question Set

New Course - 2021 (Three Years Diploma in Engineering All Programmes)

Year - II

Semester - I

Subject: Engineering Mathematics - III

F.M. : 80

P.M. : 32

Time: 3 hrs

Group 'A'

[(7 × 2) × 2 = 28]

Attempt All questions.

1. (a) Find the derivative of $\tan^{-1} \left(\frac{2x}{1-x^2} \right)$.
(b) Find the slope of the curve $y = 2x + x^2$ at (0, 1).
2. (a) Define stationary points in a curve and find the points of stationary in the curve $5x^3 - 135x + 22$.
(b) State L'Hospital's Theorem and use it to evaluate $\lim_{x \rightarrow 3} \frac{x^3 - 27}{x - 3}$.
3. (a) Define partial derivative of a function and find f_x for $f(x, y) = x^2 y^2 + 2y$
(b) Find $\frac{\partial y}{\partial x}$ if $y = x^2 - 5x + 7$ and $x = 9rs + 2t^2 s^2$.
4. (a) Test whether the given function is even, odd or neither where $f(x) = \sqrt{1+x^2} - \sqrt{1-x^2}$
(b) Test the periodicity and find the smallest period of $f(x) = \cos 2x$.
5. (a) Find the area bounded by the curve $y = 3x^2$, x-axis and the ordinates $x = 1$, $x = 2$.
(b) Determine the order and degree of the differential equation:
$$\frac{d^2 y}{dx^2} + 2 \left(\frac{dy}{dx} \right)^2 = xy.$$
6. (a) Solve by separation of variable method of: $(1+x)ydx + (1+y)x dy = 0$.
(b) Define exact differentiate equation and test the exactness of $(x+y-1)dx + (x-y-z)dy = 0$
7. (a) Define the Langrange's Linear Partial Differential Equation with an example.
(b) Define Fourier Series and write the formulae to find the Fourier coefficients.

Group 'B'

[13 × 4 = 52]

Attempt All questions.

8. Find the equation of tangent and normal to the curve $x^2 - y^2 = 7$ at (1, 2).
OR
If the area of a circle increases at a uniform rate prove that the rate of increase of the perimeter varies inversely as the radius.
9. Find the local maxima and local minima of the function below and also find the point of inflection $f(x) = 2x^3 - 15x^2 + 36x + 11$.
10. Evaluate: $\lim_{x \rightarrow 0} \frac{x \cos x - \log(1+x)}{x^2}$
11. If $u = \log(x^2 + y^2)$ then show that $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$.

12. State the Euler's theorem of homogeneous function and show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u \text{ for } u = \tan^{-1} \frac{x^3 + y^3}{x - y}.$$

13. Integrate the standard integral : $\int \frac{dx}{3 \sin x + 4 \cos x}.$

14. Find the area of the plane region bounded by the x-axis the curve $y = e^{3x}$ and the ordinates $x = 0, x = b$ using the limit of sum.

15. Find the area of the circle $x^2 + y^2 = 16$.

16. Solve the differential equation $x \frac{dy}{dx} = y - x \tan \frac{y}{x}.$

17. Show that the differential equation exact and solve:

$$(5x^4 + 3x^2y^2 - 2xy^3) dx + (2x^3y - 3x^2y^2 - 5y^4) dy = 0$$

18. Define linear differential equation and solve: $x \frac{dy}{dx} + y = x^4.$

19. Solve: $(xz - ny)p + (nx - lz)q = ly - mx.$

OR

If the portion of a tangent to a curve included between the co-ordinate axes is bisected by the point of contact then show that the curve is rectangular hyperbola.

20. Find the Fourier series of $f(x)$ which is defined by $f(x) = \begin{cases} 1 & \text{for } -\pi < x < 0 \\ -1 & \text{for } 0 \leq x < \pi \end{cases}$

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